

*Convex Analysis and
Optimization*

Chapter 7 Solutions

Dimitri P. Bertsekas

with

Angelia Nedić and Asuman E. Ozdaglar

Massachusetts Institute of Technology

Athena Scientific, Belmont, Massachusetts

<http://www.athenasc.com>

CHAPTER 7: SOLUTION MANUAL

7.1 (Fenchel's Inequality)

Let $f : \mathfrak{R}^n \mapsto (-\infty, \infty]$ be a closed proper convex function and let g be its convex conjugate.

(a) For any $x \in \mathfrak{R}^n$ and $\lambda \in \mathfrak{R}^n$, we have

$$x'\lambda \leq f(x) + g(\lambda).$$

Furthermore, the following are equivalent:

(i) $x'\lambda = f(x) + g(\lambda).$

(ii) $\lambda \in \partial f(x).$

(iii) $x \in \partial g(\lambda).$

(b) The set of minima of f over \mathfrak{R}^n is $\partial g(0).$

(c) The set of minima of f over \mathfrak{R}^n is nonempty if $0 \in \text{ri}(\text{dom}(g))$, and it is nonempty and compact if and only if $0 \in \text{int}(\text{dom}(g))$.

Solution: (a) From the definition of g ,

$$g(\lambda) = \sup_{x \in \mathfrak{R}^n} \{x'\lambda - f(x)\},$$

we have the inequality $x'\lambda \leq f(x) + g(\lambda)$. In view of this inequality, the equality $x'\lambda = f(x) + g(\lambda)$ of (i) is equivalent to the inequality

$$x'\lambda - f(x) \geq g(\lambda) = \sup_{z \in \mathfrak{R}^n} \{z'\lambda - f(z)\},$$

or

$$x'\lambda - f(x) \geq z'\lambda - f(z), \quad \forall z \in \mathfrak{R}^n,$$

or

$$f(z) \geq f(x) + \lambda'(z - x), \quad \forall z \in \mathfrak{R}^n,$$

which is equivalent to (ii). Since f is closed, f is equal to the conjugate of g , so by using the equivalence of (i) and (ii) with the roles of f and g reversed, we obtain the equivalence of (i) and (iii).

(b) A vector x^* minimizes f if and only if $0 \in \partial f(x^*)$, which by part (a), is true if and only if $x^* \in \partial g(0)$.

(c) The result follows by combining part (b) and Prop. 4.4.2.

7.2

Let $f : \mathfrak{R}^n \mapsto (-\infty, \infty]$ be a proper convex function, and let g be its conjugate. Show that the lineality space of g is equal to the orthogonal complement of the subspace parallel to $\text{aff}(\text{dom}(f))$.

Solution: Let $f : \mathfrak{R}^n \mapsto (-\infty, \infty]$ be a proper convex function, and let g be its conjugate. Show that the lineality space of g is equal to the orthogonal complement of the subspace parallel to $\text{aff}(\text{dom}(f))$.

7.3

Let $f_i : \mathfrak{R}^n \mapsto (-\infty, \infty]$, $i = 1, \dots, m$, be proper convex functions, and let $f = f_1 + \dots + f_m$. Show that if $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is nonempty, then we have

$$g(\lambda) = \inf_{\substack{\lambda_1 + \dots + \lambda_m = \lambda \\ \lambda_i \in \mathfrak{R}^n, i=1, \dots, m}} \{g_1(\lambda_1) + \dots + g_m(\lambda_m)\}, \quad \forall \lambda \in \mathfrak{R}^n,$$

where g, g_1, \dots, g_m are the conjugates of f, f_1, \dots, f_m , respectively.

Solution: Let $f_i : \mathfrak{R}^n \mapsto (-\infty, \infty]$, $i = 1, \dots, m$, be proper convex functions, and let $f = f_1 + \dots + f_m$. Show that if $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is nonempty, then we have

$$g(\lambda) = \inf_{\substack{\lambda_1 + \dots + \lambda_m = \lambda \\ \lambda_i \in \mathfrak{R}^n, i=1, \dots, m}} \{g_1(\lambda_1) + \dots + g_m(\lambda_m)\}, \quad \forall \lambda \in \mathfrak{R}^n,$$

where g, g_1, \dots, g_m are the conjugates of f, f_1, \dots, f_m , respectively.

7.4 (Finiteness of the Optimal Dual Value)

Consider the problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in X, \quad g_j(x) \leq 0, \quad j = 1, \dots, r, \end{aligned}$$

where X is a convex set, and f and g_j are convex over X . Assume that the problem has at least one feasible solution. Show that the following are equivalent.

- (i) The dual optimal value $q^* = \sup_{\mu \geq 0} q(\mu)$ is finite.
- (ii) The primal function p is proper.
- (iii) The set

$$M = \{(u, w) \in \mathfrak{R}^{r+1} \mid \text{there is an } x \in X \text{ such that } g(x) \leq u, f(x) \leq w\}$$

does not contain a vertical line.

Solution: Consider the function \tilde{q} given by

$$\tilde{q}(\mu) = \begin{cases} q(\mu) & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

and note that $-\tilde{q}$ is closed and convex, and that by the calculation of Example 7.1.6, we have

$$\tilde{q}(\mu) = \inf_{u \in \mathfrak{R}^r} \{p(u) + \mu' u\}, \quad \forall \mu \in \mathfrak{R}^r. \quad (1)$$

Since $\tilde{q}(\mu) \leq p(0)$ for all $\mu \in \mathfrak{R}^r$, given the feasibility of the problem [i.e., $p(0) < \infty$], we see that q^* is finite if and only if $-\tilde{q}$ is proper. From Eq. (1), $-\tilde{q}$ is the conjugate of $p(-u)$, and by the Conjugacy Theorem [Prop. 7.1.1(b)], $-\tilde{q}$ is proper if and only if p is proper. Hence, (i) is equivalent to (ii).

We note that the epigraph of p is the closure of M . Hence, given the feasibility of the problem, (ii) is equivalent to the closure of M not containing a vertical line. Since M is convex, its closure does not contain a line if and only if M does not contain a line (since the closure and the relative interior of M have the same recession cone). Hence (ii) is equivalent to (iii).

7.5 (General Perturbations and Min Common/Max Crossing Duality)

Let $F : \mathfrak{R}^{n+m} \mapsto (-\infty, \infty]$ be a proper function, and let $G : \mathfrak{R}^{n+m} \mapsto (-\infty, \infty]$ be its conjugate. Let also p be the function defined by

$$p(u) = \inf_{x \in \mathfrak{R}^n} F(x, u), \quad u \in \mathfrak{R}^m.$$

Consider the min common/max crossing framework for the set $M = \text{epi}(p)$, and the cost function of the max crossing problem, $q(\lambda) = \inf_{(u,w) \in M} \{w + \lambda' u\}$.

(a) Show that q and the conjugate h of p satisfy

$$h(\lambda) = G(0, \lambda), \quad q(\lambda) = -G(0, -\lambda), \quad \forall \lambda \in \mathfrak{R}^m.$$

Show also that these relations generalize Example 7.1.5.

(b) Consider the alternative min common/max crossing framework where

$$M = \{(u, w) \mid \text{there is an } x \text{ such that } F(x, u) \leq w\}.$$

Show that the optimal values of the corresponding min common and max crossing problems are the same as those corresponding to $M = \text{epi}(p)$.

(c) Show that with $F(x, u) = f_1(x) - f_2(Qx + u)$, the min common/max crossing framework corresponds to the Fenchel duality framework. What are the forms of F that correspond to the minimax and constrained optimization frameworks of Sections 2.6.1 and 6.1?

Solution: (a) We have

$$\begin{aligned}
 h(\lambda) &= \sup_u \{\lambda' u - p(u)\} \\
 &= \sup_u \{\lambda' u - \inf_x F(x, u)\} \\
 &= \sup_{x,u} \{\lambda' u - F(x, u)\} \\
 &= G(0, \lambda).
 \end{aligned}$$

Also

$$\begin{aligned}
 q(\lambda) &= \inf_{(u,w) \in M} \{w + \lambda' u\} \\
 &= \inf_{x,u} \{F(x, u) + \lambda' u\} \\
 &= -\sup_{x,u} \{-\lambda' u - F(x, u)\} \\
 &= -G(0, -\lambda).
 \end{aligned}$$

Consider the constrained minimization problem of Example 7.1.5:

$$\begin{aligned}
 &\text{minimize } f(x) \\
 &\text{subject to } x \in X, \quad g(x) \leq 0,
 \end{aligned}$$

and define

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in X \text{ and } g(x) \leq u, \\ \infty & \text{otherwise.} \end{cases}$$

Then p is the primal function of the constrained minimization problem. Consider now $q(\lambda)$, the cost function of the max crossing problem corresponding to M . For $\lambda \geq 0$, $q(\lambda)$ is equal to the dual function value of the constrained optimization problem, and otherwise $q(\lambda)$ is equal to $-\infty$. Thus, the relations $h(\lambda) = G(0, \lambda)$ and $q(\lambda) = -G(0, -\lambda)$ proved earlier, show the relation proved in Example 7.1.5, i.e., that $q(\lambda) = -h(-\lambda)$.

(b) Let

$$M = \{(u, w) \mid \text{there is an } x \text{ such that } F(x, u) \leq w\}.$$

Then the corresponding min common value is

$$\inf_{\{(x,w) \mid F(x,0) \leq w\}} w = \inf_x F(x, 0) = p(0).$$

Since $p(0)$ is the min common value corresponding to $\text{epi}(p)$, the min common values corresponding to the two choices for M are equal. Similarly, we show that the cost functions of the max crossing problem corresponding to the two choices for M are equal.

(c) If $F(x, u) = f_1(x) - f_2(Qx + u)$, we have

$$p(u) = \inf_x \{f_1(x) - f_2(Qx + u)\},$$

so $p(0)$, the min common value, is equal to the primal optimal value in the Fenchel duality framework. By part (a), the max crossing value is

$$q^* = \sup_{\lambda} \{-h(-\lambda)\},$$

where h is the conjugate of p . By using the change of variables $z = Qx + u$ in the following calculation, we have

$$\begin{aligned} -h(-\lambda) &= -\sup_u \{-\lambda'u - \inf_x \{f_1(x) - f_2(Qx + u)\}\} \\ &= -\sup_{z,x} \{-\lambda'(z - Qx) - f_1(x) + f_2(z)\} \\ &= g_2(\lambda) - g_1(Q\lambda), \end{aligned}$$

where g_1 and g_2 are the conjugate convex and conjugate concave functions of f_1 and f_2 , respectively:

$$g_1(\lambda) = \sup_x \{x'\lambda - f_1(x)\}, \quad g_2(\lambda) = \inf_z \{z'\lambda - f_2(z)\}.$$

Thus, no duality gap in the min common/max crossing framework [i.e., $p(0) = q^* = \sup_{\lambda} \{-h(-\lambda)\}$] is equivalent to no duality gap in the Fenchel duality framework.

The minimax framework of Section 2.6.1 (using the notation of that section) is obtained for

$$F(x, u) = \sup_{z \in Z} \{\phi(x, z) - u'z\}.$$

The constrained optimization framework of Section 6.1 (using the notation of that section) is obtained for the function

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in X, h(x) = u_1, g(x) \leq u_2, \\ \infty & \text{otherwise,} \end{cases}$$

where $u = (u_1, u_2)$.

7.6

Use Minimax Theorem III (Prop. 3.5.3) to derive the following version of the Primal Fenchel Duality Theorem: Let the functions f_1 and $-f_2$ be proper and convex. Then we have

$$\inf_{x \in \mathbb{R}^n} \{f_1(x) - f_2(x)\} = \sup_{\lambda \in \mathbb{R}^m} \{g_2(\lambda) - g_1(\lambda)\},$$

and the supremum in the right-hand side above is attained, if

$$\text{ri}(\text{dom}(f_1)) \cap \text{ri}(\text{dom}(-f_2)) \neq \emptyset.$$

Hint: In view of the results of Exercise 1.35, it is sufficient to show the above equality when f_1 and $-f_2$ are replaced by their closures.

Solution: By Exercise 1.35,

$$\text{cl } f_1 + \text{cl } (-f_2) = \text{cl } (f_1 - f_2).$$

Furthermore,

$$\inf_{x \in \mathfrak{R}^n} \text{cl } (f_1 - f_2)(x) = \inf_{x \in \mathfrak{R}^n} (f_1(x) - f_2(x)).$$

Thus, we may replace f_1 and $-f_2$ with their closures, and the result follows by applying Minimax Theorem III.

7.7 (Monotropic Programming Duality)

Consider the problem

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n f_i(x_i) \\ & \text{subject to} && x \in S, \quad x_i \in X_i, \quad i = 1, \dots, n, \end{aligned}$$

where $f_i : \mathfrak{R} \rightarrow \mathfrak{R}$ are given functions, X_i are intervals of real numbers, and S is a subspace of \mathfrak{R}^n . Assume that the problem is feasible and that its optimal value is finite.

(a) Show that a dual problem is

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n g_i(\lambda_i) \\ & \text{subject to} && \lambda \in S^\perp, \end{aligned}$$

where the functions $g_i : \mathfrak{R} \rightarrow (-\infty, \infty]$ are the conjugate convex functions

$$g_i(\lambda_i) = \sup_{x_i \in X_i} \{ \lambda_i x_i - f_i(x_i) \}, \quad i = 1, \dots, n.$$

(b) Show that the dual problem has an optimal solution and there is no duality gap under one of the following two conditions:

- (1) Each function f_i is convex over X_i and S contains a point in the relative interior of $X_1 \times \dots \times X_n$.
- (2) The intervals X_i are closed and the functions f_i are convex over the entire real line.

Solution: We apply Fenchel duality with

$$f_1(x) = \begin{cases} \sum_{i=1}^n f_i(x_i) & \text{if } x \in X_1 \times \dots \times X_n, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x \in S, \\ -\infty & \text{otherwise.} \end{cases}$$

The corresponding conjugate concave and convex functions g_2 and g_1 are

$$\inf_{x \in S} \lambda'x = \begin{cases} 0 & \text{if } \lambda \in S^\perp, \\ -\infty & \text{if } \lambda \notin S^\perp, \end{cases}$$

where S^\perp is the orthogonal subspace of S , and

$$\sup_{x_i \in X_i} \left\{ \sum_{i=1}^n (x_i \lambda_i - f_i(x_i)) \right\} = \sum_{i=1}^n g_i(\lambda_i),$$

where for each i ,

$$g_i(\lambda_i) = \sup_{x_i \in X_i} \{x_i \lambda_i - f_i(x_i)\}.$$

By the Primal Fenchel Duality Theorem (Prop. 7.2.1), the dual problem has an optimal solution and there is no duality gap if the functions f_i are convex over X_i and one of the following two conditions holds:

- (1) The subspace S contains a point in the relative interior of $X_1 \times \cdots \times X_n$.
- (2) The intervals X_i are closed (so that the Cartesian product $X_1 \times \cdots \times X_n$ is a polyhedral set) and the functions f_i are convex over the entire real line.

These conditions correspond to the two conditions for no duality gap given following Prop. 7.2.1.

7.8 (Network Optimization and Kirchhoff's Laws)

Consider a linear resistive electric network with node set \mathcal{N} and arc set \mathcal{A} . Let v_i be the voltage of node i and let x_{ij} be the current of arc (i, j) . Kirchhoff's current law says that for each node i , the total outgoing current is equal to the total incoming current

$$\sum_{\{j|(i,j) \in \mathcal{A}\}} x_{ij} = \sum_{\{j|(j,i) \in \mathcal{A}\}} x_{ji}.$$

Ohm's law says that the current x_{ij} and the voltage drop $v_i - v_j$ along each arc (i, j) are related by

$$v_i - v_j = R_{ij}x_{ij} - t_{ij},$$

where $R_{ij} \geq 0$ is a resistance parameter and t_{ij} is another parameter that is nonzero when there is a voltage source along the arc (i, j) (t_{ij} is positive if the voltage source pushes current in the direction from i to j). Consider the problem

$$\begin{aligned} & \text{minimize} && \sum_{(i,j) \in \mathcal{A}} \left(\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij} \right) \\ & \text{subject to} && \sum_{\{j|(i,j) \in \mathcal{A}\}} x_{ij} = \sum_{\{j|(j,i) \in \mathcal{A}\}} x_{ji}, \quad \forall i \in \mathcal{N}. \end{aligned} \tag{7.0}$$

Show that a set of variables $\{x_{ij} \mid (i, j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ are an optimal solution-Lagrange multiplier pair for this problem if and only if they satisfy Kirchhoff's current law and Ohm's law.

Solution: This problem is a monotropic programming problem, as considered in Exercise 7.7. For each $(i, j) \in \mathcal{A}$, the function $f_{ij}(x_{ij}) = \frac{1}{2}R_{ij}x_{ij}^2 - t_{ij}x_{ij}$ is continuously differentiable and convex over \Re . The dual problem is

$$\begin{aligned} & \text{maximize } q(v) \\ & \text{subject to no constraints on } p, \end{aligned}$$

with the dual function q given by

$$q(v) = \sum_{(i,j) \in \mathcal{A}} q_{ij}(v_i - v_j),$$

where

$$q_{ij}(v_i - v_j) = \min_{x_{ij} \in \Re} \left\{ \frac{1}{2}R_{ij}x_{ij}^2 - (v_i - v_j + t_{ij})x_{ij} \right\}.$$

Since the primal cost functions f_{ij} are real-valued and convex over the entire real line, there is no duality gap. The necessary and sufficient conditions for a set of variables $\{x_{ij} \mid (i, j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ to be an optimal solution-Lagrange multiplier pair are:

- (1) The set of variables $\{x_{ij} \mid (i, j) \in \mathcal{A}\}$ must be primal feasible, i.e., Kirchhoff's current law must be satisfied.
- (2)

$$x_{ij} \in \arg \min_{y_{ij} \in \Re} \left\{ \frac{1}{2}R_{ij}y_{ij}^2 - (v_i - v_j + t_{ij})y_{ij} \right\}, \quad \forall (i, j) \in \mathcal{A},$$

which is equivalent to Ohm's law:

$$R_{ij}x_{ij} - (v_i - v_j + t_{ij}) = 0, \quad \forall (i, j) \in \mathcal{A}.$$

Hence a set of variables $\{x_{ij} \mid (i, j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ are an optimal solution-Lagrange multiplier pair if and only if they satisfy Kirchhoff's current law and Ohm's law.

7.9 (Symmetry of Duality)

Consider the primal function

$$p(u) = \inf_{x \in X, g(x) \leq u} f(x)$$

of the problem

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } x \in X, \quad g_j(x) \leq 0, \quad j = 1, \dots, r. \end{aligned} \tag{7.1}$$

Consider also the problem

$$\begin{aligned} & \text{minimize } p(u) \\ & \text{subject to } u \in P, \quad u \leq 0, \end{aligned} \tag{7.2}$$

where P is the effective domain of p ,

$$P = \{u \mid \text{there exists } x \in X \text{ with } g(x) \leq u\}.$$

Assume that $-\infty < p(0) < \infty$.

- (a) Show that problems (7.1) and (7.2) have equal optimal values, and the same sets of geometric multipliers.
- (b) Consider the dual functions of problems (7.1) and (7.2) and show that they are equal on the positive orthant, i.e., for all $\mu \geq 0$,

$$q(\mu) = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^r \mu_j g_j(x) \right\} = \inf_{u \in P} \{p(u) + \mu' u\}.$$

- (c) Assume that p is a closed and convex function. Show that u^* is an optimal solution of problem (7.2) if and only if $-u^*$ is a geometric multiplier for the dual problem

$$\begin{aligned} & \text{maximize } q(\mu) \\ & \text{subject to } \mu \geq 0, \end{aligned}$$

in the sense that

$$q^* = \sup_{\mu \geq 0} \{q(\mu) - \mu' u^*\}.$$

Solution: (a) We have $f^* = p(0)$. Since $p(u)$ is monotonically nonincreasing, its minimal value over $u \in P$ and $u \leq 0$ is attained for $u = 0$. Hence, $f^* = p^*$, where $p^* = \inf_{u \in P, u \leq 0} p(u)$. For $\mu \geq 0$, we have

$$\begin{aligned} \inf_{x \in X} \{f(x) + \mu' g(x)\} &= \inf_{u \in P} \inf_{x \in X, g(x) \leq u} \{f(x) + \mu' g(x)\} \\ &= \inf_{u \in P} \{p(u) + \mu' u\}. \end{aligned}$$

Since $f^* = p^*$, we see that $f^* = \inf_{x \in X} \{f(x) + \mu' g(x)\}$ if and only if $p^* = \inf_{u \in P} \{p(u) + \mu' u\}$. In other words, the two problems have the same geometric multipliers.

(b) This part was proved by the preceding argument.

(c) From Example 7.1.5, we have that $-q(-\mu)$ is the conjugate convex function of p . Let us view the dual problem as the minimization problem

$$\begin{aligned} & \text{minimize } -q(-\mu) \\ & \text{subject to } \mu \leq 0. \end{aligned} \tag{1}$$

Its dual problem is obtained by forming the conjugate convex function of its primal function, which is p , based on the analysis of Example 7.1.5, and the closedness and convexity of p . Hence the dual of the dual problem (1) is

$$\begin{aligned} & \text{maximize} && -p(u) \\ & \text{subject to} && u \leq 0 \end{aligned}$$

and the optimal solutions to this problem are the geometric multipliers to problem (1).

7.10 (Second-Order Cone Programming)

Consider the problem

$$\begin{aligned} & \text{minimize} && c'x \\ & \text{subject to} && \|A_jx + b_j\| \leq e'_jx + d_j, \quad j = 1, \dots, r, \end{aligned}$$

where $x \in \mathfrak{R}^n$, and c, A_j, b_j, e_j , and d_j are given, and have appropriate dimension. Assume that the problem is feasible. Consider the equivalent problem

$$\begin{aligned} & \text{minimize} && c'x \\ & \text{subject to} && \|u_j\| \leq t_j, \quad u_j = A_jx + b_j, \quad t_j = e'_jx + d_j, \quad j = 1, \dots, r, \end{aligned} \tag{7.3}$$

where u_j and t_j are auxiliary optimization variables.

- (a) Show that problem (7.3) has cone constraints of the type described in Section 7.2.2.
- (b) Use the conic duality theory of Section 7.2.2 to show that a dual problem is given by

$$\begin{aligned} & \text{minimize} && \sum_{j=1}^r (b'_jz_j + d_jw_j) \\ & \text{subject to} && \sum_{j=1}^r (A'_jz_j + e_jw_j) = c, \quad \|z_j\| \leq w_j, \quad j = 1, \dots, r. \end{aligned} \tag{7.4}$$

Furthermore, show that there is no duality gap if either there exists a feasible solution of problem (7.3) or a feasible solution of problem (7.4) satisfying strictly all the corresponding inequality constraints.

Solution: (a) Define

$$X = \{(x, u, t) \mid x \in \mathfrak{R}^n, u_j = A_jx + b_j, t_j = e'_jx + d_j, j = 1, \dots, r\},$$

$$C = \{(x, u, t) \mid x \in \mathfrak{R}^n, \|u_j\| \leq t_j, j = 1, \dots, r\}.$$

It can be seen that X is convex and C is a cone. Therefore the modified problem can be written as

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && x \in X \cap C, \end{aligned}$$

and is a cone programming problem of the type described in Section 7.2.2.

(b) Let $(\lambda, z, w) \in \hat{C}$, where \hat{C} is the dual cone ($\hat{C} = -C^*$, where C^* is the polar cone). Then we have

$$\lambda'x + \sum_{j=1}^r z'_j u_j + \sum_{j=1}^r w_j t_j \geq 0, \quad \forall (x, u, t) \in C.$$

Since x is unconstrained, we must have $\lambda = 0$ for otherwise the above inequality will be violated. Furthermore, it can be seen that

$$\hat{C} = \{(0, z, w) \mid \|z_j\| \leq w_j, j = 1, \dots, r\}.$$

By the conic duality theory of Section 7.2.2, the dual problem is given by

$$\begin{aligned} & \text{minimize} && \sum_{j=1}^r (z'_j b_j + w_j d_j) \\ & \text{subject to} && \sum_{j=1}^r (A'_j z_j + w_j e_j) = c, \quad \|z_j\| \leq w_j, j = 1, \dots, r. \end{aligned}$$

If there exists a feasible solution of the modified primal problem satisfying strictly all the inequality constraints, then the relative interior condition $\text{ri}(X) \cap \text{ri}(C) \neq \emptyset$ is satisfied, and there is no duality gap. Similarly, if there exists a feasible solution of the dual problem satisfying strictly all the inequality constraints, there is no duality gap.

7.11 (Quadratically Constrained Quadratic Problems [LVB98])

Consider the quadratically constrained quadratic problem

$$\begin{aligned} & \text{minimize} && x' P_0 x + 2q'_0 x + r_0 \\ & \text{subject to} && x' P_i x + 2q'_i x + r_i \leq 0, \quad i = 1, \dots, p, \end{aligned}$$

where P_0, P_1, \dots, P_p are symmetric positive definite matrices. Show that the problem can be converted to one of the type described in Exercise 7.10, and derive the corresponding dual problem. *Hint*: Consider the equivalent problem

$$\begin{aligned} & \text{minimize} && \|P_0^{1/2} x + P_0^{-1/2} q_0\| \\ & \text{subject to} && \|P_i^{1/2} x + P_i^{-1/2} q_i\| \leq (r_i - q'_i P_i^{-1} q_i)^{1/2}, \quad i = 1, \dots, p. \end{aligned}$$

Solution: Since each P_i is symmetric and positive definite, we have

$$\begin{aligned} x'P_i x + 2q_i'x + r_i &= \left(P_i^{1/2}x\right)' P_i^{1/2}x + 2\left(P_i^{-1/2}q_i\right)' P_i^{1/2}x + r_i \\ &= \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i'P_i^{-1}q_i, \end{aligned}$$

for $i = 0, 1, \dots, p$. This allows us to write the original problem as

$$\begin{aligned} &\text{minimize } \|P_0^{1/2}x + P_0^{-1/2}q_0\|^2 + r_0 - q_0'P_0^{-1}q_0 \\ &\text{subject to } \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i'P_i^{-1}q_i \leq 0, \quad i = 1, \dots, p. \end{aligned}$$

By introducing a new variable x_{n+1} , this problem can be formulated in \Re^{n+1} as

$$\begin{aligned} &\text{minimize } x_{n+1} \\ &\text{subject to } \|P_0^{1/2}x + P_0^{-1/2}q_0\| \leq x_{n+1} \\ &\|P_i^{1/2}x + P_i^{-1/2}q_i\| \leq (q_i'P_i^{-1}q_i - r_i)^{1/2}, \quad i = 1, \dots, p. \end{aligned}$$

The optimal values of this problem and the original problem are equal up to a constant and a square root. The above problem is of the type described in Exercise 7.10. To see this, define $A_i = \begin{pmatrix} P_i^{1/2} & | & 0 \end{pmatrix}$, $b_i = P_i^{-1/2}q_i$, $e_i = 0$, $d_i = (q_i'P_i^{-1}q_i - r_i)^{1/2}$ for $i = 1, \dots, p$, $A_0 = \begin{pmatrix} P_0^{1/2} & | & 0 \end{pmatrix}$, $b_0 = P_0^{-1/2}q_0$, $e_0 = (0, \dots, 0, 1)$, $d_0 = 0$, and $c = (0, \dots, 0, 1)$. Its dual is given by

$$\begin{aligned} &\text{maximize } -\sum_{i=1}^p \left(q_i'P_i^{-1/2}z_i + (q_i'P_i^{-1}q_i - r_i)^{1/2} w_i \right) - q_0'P_0^{-1/2}z_0 \\ &\text{subject to } \sum_{i=0}^p P_i^{1/2}z_i = 0, \quad \|z_0\| \leq 1, \quad \|z_i\| \leq w_i, \quad i = 1, \dots, p. \end{aligned}$$

7.12 (Minimizing the Sum or the Maximum of Norms [LVB98])

Consider the problems

$$\begin{aligned} &\text{minimize } \sum_{i=1}^p \|F_i x + g_i\| \\ &\text{subject to } x \in \Re^n, \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &\text{minimize } \max_{i=1, \dots, p} \|F_i x + g_i\| \\ &\text{subject to } x \in \Re^n, \end{aligned}$$

where F_i and g_i are given matrices and vectors, respectively. Convert these problems to second-order cone programming problems (cf. Exercise 7.10) and derive the corresponding dual problems.

Solution: Consider the problem

$$\begin{aligned} & \text{minimize} \quad \sum_{i=1}^p \|F_i x + g_i\| \\ & \text{subject to} \quad x \in \mathfrak{R}^n. \end{aligned}$$

By introducing variables t_1, \dots, t_p , this problem can be expressed as a second-order cone programming problem (see Exercise 7.10):

$$\begin{aligned} & \text{minimize} \quad \sum_{i=1}^p t_i \\ & \text{subject to} \quad \|F_i x + g_i\| \leq t_i, \quad i = 1, \dots, p. \end{aligned}$$

Define

$$\begin{aligned} X &= \{(x, u, t) \mid x \in \mathfrak{R}^n, u_i = F_i x + g_i, t_i \in \mathfrak{R}, i = 1, \dots, p\}, \\ C &= \{(x, u, t) \mid x \in \mathfrak{R}^n, \|u_i\| \leq t_i, i = 1, \dots, p\}. \end{aligned}$$

Then, similar to Exercise 7.10, we have

$$-C^* = \{(0, z, w) \mid \|z_i\| \leq w_i, i = 1, \dots, p\},$$

and

$$\begin{aligned} g(0, z, w) &= \sup_{(x, u, t) \in X} \left\{ \sum_{i=1}^p z_i' u_i + \sum_{i=1}^p w_i t_i - \sum_{i=1}^p t_i \right\} \\ &= \sup_{x \in \mathfrak{R}^n, t \in \mathfrak{R}^p} \left\{ \sum_{i=1}^p z_i' (F_i x + g_i) + \sum_{i=1}^p (w_i - 1) t_i \right\} \\ &= \sup_{x \in \mathfrak{R}^n} \left\{ \left(\sum_{i=1}^p F_i' z_i \right)' x \right\} + \sup_{t \in \mathfrak{R}^p} \left\{ \sum_{i=1}^p (w_i - 1) t_i \right\} + \sum_{i=1}^p g_i' z_i \\ &= \begin{cases} \sum_{i=1}^p g_i' z_i & \text{if } \sum_{i=1}^p F_i' z_i = 0, w_i = 1, i = 1, \dots, p \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the dual problem is given by

$$\begin{aligned} & \text{maximize} \quad - \sum_{i=1}^p g_i' z_i \\ & \text{subject to} \quad \sum_{i=1}^p F_i' z_i = 0, \quad \|z_i\| \leq 1, \quad i = 1, \dots, p. \end{aligned}$$

Now, consider the problem

$$\begin{aligned} & \text{minimize} && \max_{1 \leq i \leq p} \|F_i x + g_i\| \\ & \text{subject to} && x \in \mathfrak{R}^n. \end{aligned}$$

By introducing a new variable x_{n+1} , we obtain

$$\begin{aligned} & \text{minimize} && x_{n+1} \\ & \text{subject to} && \|F_i x + g_i\| \leq x_{n+1}, \quad i = 1, \dots, p, \end{aligned}$$

or equivalently

$$\begin{aligned} & \text{minimize} && e'_{n+1} x \\ & \text{subject to} && \|A_i x + g_i\| \leq e'_{n+1} x, \quad i = 1, \dots, p, \end{aligned}$$

where $x \in \mathfrak{R}^{n+1}$, $A_i = (F_i, 0)$, and $e_{n+1} = (0, \dots, 0, 1)' \in \mathfrak{R}^{n+1}$. Evidently, this is a second-order cone programming problem. From Exercise 7.10 we have that its dual problem is given by

$$\begin{aligned} & \text{maximize} && - \sum_{i=1}^p g'_i z_i \\ & \text{subject to} && \sum_{i=1}^p \left(\begin{pmatrix} F'_i \\ 0 \end{pmatrix} z_i + e_{n+1} w_i \right) = e_{n+1}, \quad \|z_i\| \leq w_i, \quad i = 1, \dots, p, \end{aligned}$$

or equivalently

$$\begin{aligned} & \text{maximize} && - \sum_{i=1}^p g'_i z_i \\ & \text{subject to} && \sum_{i=1}^p F'_i z_i = 0, \quad \sum_{i=1}^p w_i = 1, \quad \|z_i\| \leq w_i, \quad i = 1, \dots, p. \end{aligned}$$

7.13 (Complex l_1 and l_∞ Approximation [LVB98])

Consider the complex l_1 approximation problem

$$\begin{aligned} & \text{minimize} && \|Ax - b\|_1 \\ & \text{subject to} && x \in \mathcal{C}^n, \end{aligned}$$

where \mathcal{C}^n is the set of n -dimensional vectors whose components are complex numbers. Show that it is a special case of problem (7.5) and derive the corresponding dual problem. Repeat for the complex l_∞ approximation problem

$$\begin{aligned} & \text{minimize} && \|Ax - b\|_\infty \\ & \text{subject to} && x \in \mathcal{C}^n. \end{aligned}$$

Solution:

For $v \in \mathcal{C}^p$ we have

$$\|v\|_1 = \sum_{i=1}^p |v_i| = \sum_{i=1}^p \left\| \begin{pmatrix} \mathcal{R}e(v_i) \\ \mathcal{I}m(v_i) \end{pmatrix} \right\|,$$

where $\mathcal{R}e(v_i)$ and $\mathcal{I}m(v_i)$ denote the real and the imaginary parts of v_i , respectively. Then the complex l_1 approximation problem is equivalent to

$$\begin{aligned} & \text{minimize} \quad \sum_{i=1}^p \left\| \begin{pmatrix} \mathcal{R}e(a'_i x - b_i) \\ \mathcal{I}m(a'_i x - b_i) \end{pmatrix} \right\| \\ & \text{subject to} \quad x \in \mathcal{C}^n, \end{aligned} \tag{1}$$

where a'_i is the i -th row of A (A is a $p \times n$ matrix). Note that

$$\begin{pmatrix} \mathcal{R}e(a'_i x - b_i) \\ \mathcal{I}m(a'_i x - b_i) \end{pmatrix} = \begin{pmatrix} \mathcal{R}e(a'_i) & -\mathcal{I}m(a'_i) \\ \mathcal{I}m(a'_i) & \mathcal{R}e(a'_i) \end{pmatrix} \begin{pmatrix} \mathcal{R}e(x) \\ \mathcal{I}m(x) \end{pmatrix} - \begin{pmatrix} \mathcal{R}e(b_i) \\ \mathcal{I}m(b_i) \end{pmatrix}.$$

By introducing new variables $y = (\mathcal{R}e(x'), \mathcal{I}m(x'))'$, problem (1) can be rewritten as

$$\begin{aligned} & \text{minimize} \quad \sum_{i=1}^p \|F_i y + g_i\| \\ & \text{subject to} \quad y \in \mathfrak{R}^{2n}, \end{aligned}$$

where

$$F_i = \begin{pmatrix} \mathcal{R}e(a'_i) & -\mathcal{I}m(a'_i) \\ \mathcal{I}m(a'_i) & \mathcal{R}e(a'_i) \end{pmatrix}, \quad g_i = - \begin{pmatrix} \mathcal{R}e(b_i) \\ \mathcal{I}m(b_i) \end{pmatrix}. \tag{2}$$

According to Exercise 7.12, the dual problem is given by

$$\begin{aligned} & \text{maximize} \quad \sum_{i=1}^p (\mathcal{R}e(b_i), \mathcal{I}m(b_i)) z_i \\ & \text{subject to} \quad \sum_{i=1}^p \begin{pmatrix} \mathcal{R}e(a'_i) & \mathcal{I}m(a'_i) \\ -\mathcal{I}m(a'_i) & \mathcal{R}e(a'_i) \end{pmatrix} z_i = 0, \quad \|z_i\| \leq 1, \quad i = 1, \dots, p, \end{aligned}$$

where $z_i \in \mathfrak{R}^{2n}$ for all i .

For $v \in \mathcal{C}^p$ we have

$$\|v\|_\infty = \max_{1 \leq i \leq p} |v_i| = \max_{1 \leq i \leq p} \left\| \begin{pmatrix} \mathcal{R}e(v_i) \\ \mathcal{I}m(v_i) \end{pmatrix} \right\|.$$

Therefore the complex l_∞ approximation problem is equivalent to

$$\begin{aligned} & \text{minimize} \quad \max_{1 \leq i \leq p} \left\| \begin{pmatrix} \mathcal{R}e(a'_i x - b_i) \\ \mathcal{I}m(a'_i x - b_i) \end{pmatrix} \right\| \\ & \text{subject to} \quad x \in \mathcal{C}^n. \end{aligned}$$

By introducing new variables $y = (\mathcal{R}e(x'), \mathcal{I}m(x'))'$, this problem can be rewritten as

$$\begin{aligned} & \text{minimize } \max_{1 \leq i \leq p} \|F_i y + g_i\| \\ & \text{subject to } y \in \mathfrak{R}^{2n}, \end{aligned}$$

where F_i and g_i are given by Eq. (2). From Exercise 7.12, it follows that the dual problem is

$$\begin{aligned} & \text{maximize } \sum_{i=1}^p (\mathcal{R}e(b_i), \mathcal{I}m(b_i)) z_i \\ & \text{subject to } \sum_{i=1}^p \begin{pmatrix} \mathcal{R}e(a'_i) & -\mathcal{I}m(a'_i) \\ \mathcal{I}m(a'_i) & \mathcal{R}e(a'_i) \end{pmatrix} z_i = 0, \quad \sum_{i=1}^p w_i = 1, \quad \|z_i\| \leq w_i, \\ & \hspace{15em} i = 1, \dots, p, \end{aligned}$$

where $z_i \in \mathfrak{R}^2$ for all i .

7.14

Consider the case where in Prop. 7.3.1 the function P has the form

$$P(z) = \sum_{j=1}^r P_j(z_j),$$

where $P_j : \mathfrak{R} \mapsto \mathfrak{R}$ are convex real-valued functions satisfying

$$P_j(z_j) = 0, \quad \forall z_j \leq 0, \quad P_j(z_j) > 0, \quad \forall z_j > 0.$$

Show that the conditions 7.22 and 7.23 of Prop. 7.3.1 are equivalent to

$$\mu_j^* \leq \lim_{z_j \downarrow 0} \frac{P_j(z_j)}{z_j}, \quad \forall j = 1, \dots, r,$$

and

$$\mu_j^* < \lim_{z_j \downarrow 0} \frac{P_j(z_j)}{z_j}, \quad \forall j = 1, \dots, r,$$

respectively.

Solution: The condition $u' \mu^* \leq P(u)$ for all $u \in \mathfrak{R}^r$ can be written as

$$\sum_{j=1}^r u_j \mu_j^* \leq \sum_{j=1}^r P_j(u_j), \quad \forall u = (u_1, \dots, u_r),$$

and is equivalent to

$$u_j \mu_j^* \leq P_j(u_j), \quad \forall u_j \in \mathfrak{R}, \quad \forall j = 1, \dots, r.$$

In view of the requirement that P_j is convex with $P_j(u_j) = 0$ for $u_j \leq 0$, and $P_j(u_j) > 0$ for all $u_j > 0$, it follows that the condition $u_j \mu_j^* \leq P_j(u_j)$ for all $u_j \in \mathfrak{R}$, is equivalent to $\mu_j^* \leq \lim_{z_j \downarrow 0} (P_j(z_j)/z_j)$. Similarly, the condition $u_j \mu_j^* < P_j(u_j)$ for all $u_j \in \mathfrak{R}$, is equivalent to $\mu_j^* < \lim_{z_j \downarrow 0} (P_j(z_j)/z_j)$.

7.15 [Ber99b]

For a vector $y \in \mathfrak{R}^n$, let $d(y)$ be the optimal value of the projection problem

$$\begin{aligned} & \text{minimize} && \|y - x\| \\ & \text{subject to} && x \in X, \quad g_j(x) \leq 0, \quad j = 1, \dots, r, \end{aligned}$$

where X is a convex subset of \mathfrak{R}^n , and the functions $g_j : \mathfrak{R}^n \mapsto \mathfrak{R}$ are convex over X . Assuming that the problem is feasible, show that there exists a constant c such that

$$d(y) \leq c \|(g(y))^+\|, \quad \forall y \in X,$$

if and only if the projection problem has a geometric multiplier $\mu^*(y)$ such that the set $\{\mu^*(y) \mid y \in X\}$ is bounded.

Solution: Following [Ber99b], we address the problem by embedding it in a broader class of problems. Let Y be a subset of \mathfrak{R}^n , let y be a parameter vector taking values in Y , and consider the parametric program

$$\begin{aligned} & \text{minimize} && f(x, y) \\ & \text{subject to} && x \in X, \quad g_j(x, y) \leq 0, \quad j = 1, \dots, r, \end{aligned} \tag{1}$$

where X is a convex subset of \mathfrak{R}^n , and for each $y \in Y$, $f(\cdot, y)$ and $g_j(\cdot, y)$ are real-valued functions that are convex over X . We assume that for each $y \in Y$, this program has a finite optimal value, denoted by $f^*(y)$. Let $c > 0$ denote a penalty parameter and assume that the penalized problem

$$\begin{aligned} & \text{minimize} && f(x, y) + c\|g^+(x, y)\| \\ & \text{subject to} && x \in X \end{aligned} \tag{2}$$

has a finite optimal value, thereby coming under the framework of Section 7.3. By Prop. 7.3.1, we have

$$f^*(y) = \inf_{x \in X} \{f(x, y) + c\|g^+(x, y)\|\}, \quad \forall y \in Y, \tag{3}$$

if and only if

$$u' \mu^*(y) \leq c\|u^+\|, \quad \forall u \in \mathfrak{R}^r, \forall y \in Y,$$

for some geometric multiplier $\mu^*(y)$.

It is seen that Eq. (3) is equivalent to the bound

$$f^*(y) \leq f(x, y) + c\|g^+(x, y)\|, \quad \forall x \in X, \forall y \in Y, \tag{4}$$

so this bound holds if and only if there exists a uniform bounding constant $c > 0$ such that

$$u' \mu^*(y) \leq c\|u^+\|, \quad \forall u \in \mathfrak{R}^r, \forall y \in Y. \tag{5}$$

Thus the bound (4), holds if and only if for every $y \in Y$, it is possible to select a geometric multiplier $\mu^*(y)$ of the parametric problem (1) such that the set $\{\mu^*(y) \mid y \in Y\}$ is bounded.

Let us now specialize the preceding discussion to the parametric program

$$\begin{aligned} & \text{minimize } f(x, y) = \|y - x\| \\ & \text{subject to } x \in X, \quad g_j(x) \leq 0, \quad j = 1, \dots, r, \end{aligned} \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm, X is a convex subset of \mathfrak{R}^n , and g_j are convex over X . This is the projection problem of the exercise. Let us take $Y = X$. If c satisfies Eq. (5), the bound (4) becomes

$$d(y) \leq \|y - x\| + c\|(g(x))^+\|, \quad \forall x \in X, \forall y \in X,$$

and (by taking $x = y$) implies the bound

$$d(y) \leq c\|(g(y))^+\|, \quad \forall y \in X. \tag{7}$$

This bound holds if a geometric multiplier $\mu^*(y)$ of the projection problem (6) can be found such that Eq. (5) holds. We will now show the reverse assertion.

Indeed, assume that for some c , Eq. (7) holds, and to arrive at a contradiction, assume that there exist $x \in X$ and $y \in Y$ such that

$$d(y) > \|y - x\| + c\|(g(x))^+\|.$$

Then, using Eq. (7), we obtain

$$d(y) > \|y - x\| + d(x).$$

From this relation and the triangle inequality, it follows that

$$\begin{aligned} \inf_{z \in X, g(z) \leq 0} \|y - z\| &> \|y - x\| + \inf_{z \in X, g(z) \leq 0} \|x - z\| \\ &= \inf_{z \in X, g(z) \leq 0} \{\|y - x\| + \|x - z\|\} \\ &\geq \inf_{z \in X, g(z) \leq 0} \|y - z\|, \end{aligned}$$

which is a contradiction. Thus Eq. (7) implies that we have

$$d(y) \leq \|y - x\| + c\|(g(x))^+\|, \quad \forall x \in X, \forall y \in X.$$

Using Prop. 7.3.1, this implies that there exists a geometric multiplier $\mu^*(y)$ such that

$$u' \mu^*(y) \leq c\|u^+\|, \quad \forall u \in \mathfrak{R}^r, \forall y \in X.$$

This in turn implies the boundedness of the set $\{\mu^*(y) \mid y \in X\}$.