Convex Optimization Theory

Chapter 4

Exercises and Solutions: Extended Version

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CHAPTER 4: EXERCISES AND SOLUTIONS†

4.1 (Augmented Lagrangian Duality for Equality Constraints)

Construct an augmented Lagrangian framework and derive the dual function similar to the one of Section 4.2.4 for the case of the equality constraints, i.e. when we have the constraint $Ex = d$ in place of $g(x) \leq 0$, where E is an $m \times n$ matrix and $d \in \Re^m$.

Solution: The dual function has the form

$$q_c(\mu) = \inf_{x \in X} \big\{ f(x) + \mu'(Ex - d) + \frac{c}{2}\|Ex - d\|^2 \big\}.$$

4.2

In the context of Section 4.2.2, let $F(x, u) = f_1(x) + f_2(Ax + u)$, where A is an $m \times n$ matrix, and $f_1 : \Re^n \mapsto (-\infty, \infty]$ and $f_2 : \Re^m \mapsto (-\infty, \infty]$ are closed convex functions. Show that the dual function is

$$q(\mu) = -f_1^\star(A'\mu) - f_2^\star(-\mu),$$

where $f_1^\star$ and $f_2^\star$ are the conjugate functions of f_1 and f_2, respectively. *Note:* This is the Fenchel duality framework discussed in Section 5.3.5.

Solution: From Section 4.2.1, the dual function is

$$q(\mu) = -p^\star(-\mu),$$

where $p^\star$ is the conjugate of the function

$$p(u) = \inf_{x \in \Re^n} F(x, u).$$

† This set of exercises will be periodically updated as new exercises are added. Many of the exercises and solutions given here were developed as part of my earlier convex optimization book [BNO03] (coauthored with Angelia Nedić and Asuman Ozdaglar), and are posted on the internet of that book's web site. The contribution of my coauthors in the development of these exercises and their solutions is gratefully acknowledged. Since some of the exercises and/or their solutions have been modified and also new exercises have been added, all errors are my sole responsibility.

By using the change of variables $z = Ax + u$ in the following calculation, we have

$$\begin{aligned} p^\star(-\mu) &= -\sup_u \big\{ -\mu' u - \inf_x \big\{ f_1(x) + f_2(Ax + u) \big\} \big\} \\ &= \sup_{z,x} \big\{ -\mu'(z - Ax) - f_1(x) - f_2(z) \big\} \\ &= f_1^\star(A'\mu) + f_2^\star(-\mu), \end{aligned}$$

where $f_1^\star$ and $f_2^\star$ are the conjugate functions of f_1 and f_2, respectively. Thus,

$$q(\mu) = -f_1^\star(A'\mu) - f_2^\star(-\mu).$$

4.3 (An Example of Lagrangian Duality)

Consider the problem

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to} \;\; x \in X, \qquad e_i' x = d_i, \quad i = 1, \ldots, m, \end{aligned} \tag{4.1}$$

where $f : \Re^n \mapsto \Re$ is a convex function, X is a nonempty convex set, and e_i and d_i are given vectors and scalars, respectively. Consider the min common/max crossing framework where M is the subset of $\Re^{m+1}$ given by

$$M = \Big\{ \big(e_1' x - d_1, \ldots, e_m' x - d_m, f(x) \big) \mid x \in X \Big\},$$

and assume that $w^* < \infty$.

(a) Show that w^* is equal to the optimal value of problem (4.1), and that the max crossing problem is to maximize $q(\mu)$ given by

$$q(\mu) = \inf_{x \in X} \left\{ f(x) + \sum_{i=1}^m \mu_i (e_i' x - d_i) \right\}.$$

(b) Show that the corresponding set $\overline{M}$ is convex.

(c) Show that if X is compact, then $q^* = w^*$.

(d) Show that if there exists a vector $\overline{x} \in \text{ri}(X)$ such that $e_i' \overline{x} = d_i$ for all $i = 1, \ldots, m$, then $q^* = w^*$ and the max crossing problem has an optimal solution.

Solution: (a) It is easily seen that w^* is the minimal value of $f(x)$ subject to $x \in \text{conv}(X)$ and $e_i' x = d_i$, $i = 1, \ldots, m$. The corresponding max crossing problem is given by

$$q^* = \sup_{\mu \in \Re^m} q(\mu),$$

where $q(\mu)$ is given by

$$q(\mu) = \inf_{(u,w)\in M} \{w + \mu' u\} = \inf_{x\in X} \left\{ f(x) + \sum_{i=1}^{m} \mu_i (e_i' x - d_i) \right\}.$$

(b) Consider the set

$$\overline{M} = \Big\{ \big(u_1, \ldots, u_m, w\big) \mid \text{there exists } x \in X \text{ such that } e_i' x - d_i = u_i,\ \forall\, i,\ f(x) \le w \Big\}.$$

We show that $\overline{M}$ is convex. To this end, we consider vectors $(u, w) \in \overline{M}$ and $(\tilde{u}, \tilde{w}) \in \overline{M}$, and we show that their convex combinations lie in $\overline{M}$. The definition of $\overline{M}$ implies that for some $x \in X$ and $\tilde{x} \in X$, we have

$$f(x) \le w, \qquad e_i' x - d_i = u_i, \quad i = 1, \ldots, m,$$
$$f(\tilde{x}) \le \tilde{w}, \qquad e_i' \tilde{x} - d_i = \tilde{u}_i, \quad i = 1, \ldots, m.$$

For any $\alpha \in [0, 1]$, we multiply these relations with α and 1-α, respectively, and add. By using the convexity of f, we obtain

$$f\big(\alpha x + (1-\alpha)\tilde{x}\big) \le \alpha f(x) + (1-\alpha) f(\tilde{x}) \le \alpha w + (1-\alpha)\tilde{w},$$

$$e_i'\big(\alpha x + (1-\alpha)\tilde{x}\big) - d_i = \alpha u_i + (1-\alpha)\tilde{u}_i, \quad i = 1, \ldots, m.$$

In view of the convexity of X, we have $\alpha x + (1-\alpha)\tilde{x} \in X$, so these equations imply that the convex combination of (u, w) and $(\tilde{u}, \tilde{w})$ belongs to $\overline{M}$, thus proving that $\overline{M}$ is convex.

(c) We prove this result by showing that all the assumptions of Min Common/Max Crossing Theorem I are satisfied. By assumption, w^* is finite. It follows from part (b) that the the set $\overline{M}$ is convex. Therefore, we only need to show that for every sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$, there holds $w^* \le \liminf_{k\to\infty} w_k$. Consider a sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$. Since X is compact and f is convex by assumption (which implies that f is continuous by Prop. 1.4.6), it follows from Prop. 1.1.9(c) that set M is compact. Hence, the sequence $\big\{(u_k, w_k)\big\}$ has a subsequence that converges to some $(0, \overline{w}) \in M$. Assume without loss of generality that $\big\{(u_k, w_k)\big\}$ converges to $(0, \overline{w})$. Since $(0, \overline{w}) \in M$, we get

$$w^* = \inf_{(0,w)\in M} w \le \overline{w} = \liminf_{k\to\infty} w_k,$$

proving the desired result, and thus showing that $q^* = w^*$.

(d) We prove this result by showing that all the assumptions of Min Common/Max Crossing Theorem II are satisfied. By assumption, w^* is finite.

It follows from part (b) that the set $\overline{M}$ is convex. Therefore, we only need to show that the set

$$D = \big\{(e_1'x - d_1, \ldots, e_m'x - d_m) \mid x \in X\big\}$$

contains the origin in its relative interior. The set D can equivalently be written as

$$D = E \cdot X - d,$$

where E is a matrix, whose rows are the vectors e_i', $i = 1, \ldots, m$, and d is a vector with entries equal to d_i, $i = 1, \ldots, m$. By Prop. 1.3.6, it follows that

$$\mathrm{ri}(D) = E \cdot \mathrm{ri}(X) - d.$$

Hence the assumption that there exists a vector $\overline{x} \in \mathrm{ri}(X)$ such that $E\overline{x} - d = 0$ implies that $0 \in \mathrm{ri}(D)$, thus showing that $q^* = w^*$ and that the max crossing problem has an optimal solution.

4.4 (Lagrangian Duality and Compactness of the Constraint Set)

Consider the problem of Exercise 4.3, but assume that f is linear and X is compact (instead of f and X being convex). Show that q^* is equal to the minimal value of $f(x)$ subject to $x \in \mathrm{conv}(X)$ and $e_i'x = d_i$, $i = 1, \ldots, m$. *Hint*: Show that

$$\mathrm{conv}(M) = \Big\{\big(e_1'x - d_1, \ldots, e_m'x - d_m, f(x)\big) \;\Big|\; x \in \mathrm{conv}(X)\Big\},$$

and use Exercise 4.3(c).

Solution: Clearly, the max crossing values corresponding to M and $\mathrm{conv}(M)$ are equal [this is true generically, since closed halfspaces containing M also contain $\mathrm{conv}(M)$]. The expression for $\mathrm{conv}(M)$ in the hint follows from the linearity of f. Thus, the min common/max crossing framework for $\mathrm{conv}(M)$ corresponds to the problem of minimizing $f(x)$ subject to $x \in \mathrm{conv}(X)$ and $e_i'x = d_i$, $i = 1, \ldots, m$. Since M is compact, $\mathrm{conv}(M)$ is also compact, and the result follows from Exercise 4.3(c).