1

Sample Space and Probability

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"Probability" is a very useful concept, but can be interpreted in a number of ways. As an illustration, consider the following.

A patient is admitted to the hospital and a potentially life-saving drug is administered. The following dialog takes place between the nurse and a concerned relative.

RELATIVE: Nurse, what is the probability that the drug will work?
NURSE: I hope it works, we'll know tomorrow.
RELATIVE: Yes, but what is the probability that it will?
NURSE: Each case is different, we have to wait.
RELATIVE: But let's see, out of a hundred patients that are treated under similar conditions, how many times would you expect it to work?
NURSE (somewhat annoyed): I told you, every person is different, for some it works, for some it doesn't.
RELATIVE (insisting): Then tell me, if you had to bet whether it will work or not, which side of the bet would you take?
NURSE (cheering up for a moment): I'd bet it will work.
RELATIVE (somewhat relieved): OK, now, would you be willing to lose two dollars if it doesn't work, and gain one dollar if it does?
NURSE (exasperated): What a sick thought! You are wasting my time!

In this conversation, the relative attempts to use the concept of probability to discuss an **uncertain** situation. The nurse's initial response indicates that the meaning of "probability" is not uniformly shared or understood, and the relative tries to make it more concrete. The first approach is to define probability in terms of **frequency of occurrence**, as a percentage of successes in a moderately large number of similar situations. Such an interpretation is often natural. For example, when we say that a perfectly manufactured coin lands on heads "with probability 50%," we typically mean "roughly half of the time." But the nurse may not be entirely wrong in refusing to discuss in such terms. What if this was an experimental drug that was administered for the very first time in this hospital or in the nurse's experience?

While there are many situations involving uncertainty in which the frequency interpretation is appropriate, there are other situations in which it is not. Consider, for example, a scholar who asserts that the Iliad and the Odyssey were composed by the same person, with probability 90%. Such an assertion conveys some information, but not in terms of frequencies, since the subject is a one-time event. Rather, it is an expression of the scholar's **subjective belief**. One might think that subjective beliefs are not interesting, at least from a mathematical or scientific point of view. On the other hand, people often have to make choices in the presence of uncertainty, and a systematic way of making use of their beliefs is a prerequisite for successful, or at least consistent, decision making.

In fact, the choices and actions of a rational person can reveal a lot about the inner-held subjective probabilities, even if the person does not make conscious use of probabilistic reasoning. Indeed, the last part of the earlier dialog was an attempt to infer the nurse's beliefs in an indirect manner. Since the nurse was willing to accept a one-for-one bet that the drug would work, we may infer that the probability of success was judged to be at least 50%. Had the nurse accepted the last proposed bet (two-for-one), this would have indicated a success probability of at least 2/3.

Rather than dwelling further on philosophical issues about the appropriateness of probabilistic reasoning, we will simply take it as a given that the theory of probability is useful in a broad variety of contexts, including some where the assumed probabilities only reflect subjective beliefs. There is a large body of successful applications in science, engineering, medicine, management, etc., and on the basis of this empirical evidence, probability theory is an extremely useful tool.

Our main objective in this book is to develop the art of describing uncertainty in terms of probabilistic models, as well as the skill of probabilistic reasoning. The first step, which is the subject of this chapter, is to describe the generic structure of such models and their basic properties. The models we consider assign probabilities to collections (sets) of possible outcomes. For this reason, we must begin with a short review of set theory.

1.1 SETS

Probability makes extensive use of set operations, so let us introduce at the outset the relevant notation and terminology.

A **set** is a collection of objects, which are the **elements** of the set. If S is a set and x is an element of S, we write $x \in S$. If x is not an element of S, we write $x \notin S$. A set can have no elements, in which case it is called the **empty set**, denoted by $\emptyset$.

Sets can be specified in a variety of ways. If S contains a finite number of elements, say $x_1, x_2, \ldots, x_n$, we write it as a list of the elements, in braces:

$$S = \{x_1, x_2, \ldots, x_n\}.$$

For example, the set of possible outcomes of a die roll is $\{1, 2, 3, 4, 5, 6\}$, and the set of possible outcomes of a coin toss is $\{H, T\}$, where H stands for "heads" and T stands for "tails."

If S contains infinitely many elements $x_1, x_2, \ldots$, which can be enumerated in a list (so that there are as many elements as there are positive integers) we write

$$S = \{x_1, x_2, \ldots\},$$

and we say that S is **countably infinite**. For example, the set of even integers can be written as $\{0, 2, -2, 4, -4, \ldots\}$, and is countably infinite.

Alternatively, we can consider the set of all x that have a certain property P, and denote it by

$$\{x \mid x \text{ satisfies } P\}.$$

(The symbol "|" is to be read as "such that.") For example, the set of even integers can be written as $\{k \mid k/2 \text{ is integer}\}$. Similarly, the set of all scalars x in the interval $[0, 1]$ can be written as $\{x \mid 0 \leq x \leq 1\}$. Note that the elements x of the latter set take a continuous range of values, and cannot be written down in a list (a proof is sketched in the end-of-chapter problems); such a set is said to be **uncountable**.

If every element of a set S is also an element of a set T, we say that S is a **subset** of T, and we write $S \subset T$ or $T \supset S$. If $S \subset T$ and $T \subset S$, the two sets are **equal**, and we write $S = T$. It is also expedient to introduce a **universal set**, denoted by Ω, which contains all objects that could conceivably be of interest in a particular context. Having specified the context in terms of a universal set Ω, we only consider sets S that are subsets of Ω.

Set Operations

The **complement** of a set S, with respect to the universe Ω, is the set $\{x \in \Omega \mid x \notin S\}$ of all elements of Ω that do not belong to S, and is denoted by S^c. Note that $\Omega^c = \emptyset$.

The **union** of two sets S and T is the set of all elements that belong to S or T (or both), and is denoted by $S \cup T$. The **intersection** of two sets S and T is the set of all elements that belong to both S and T, and is denoted by $S \cap T$. Thus,

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\},$$

and

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}.$$

In some cases, we will have to consider the union or the intersection of several, even infinitely many sets, defined in the obvious way. For example, if for every positive integer n, we are given a set S_n, then

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \cdots = \{x \mid x \in S_n \text{ for some } n\},$$

and

$$\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \cdots = \{x \mid x \in S_n \text{ for all } n\}.$$

Two sets are said to be disjoint if their intersection is empty. More generally, several sets are said to be **disjoint** if no two of them have a common element. A collection of sets is said to be a **partition** of a set S if the sets in the collection are disjoint and their union is S.

If x and y are two objects, we use (x, y) to denote the **ordered pair** of x and y. The set of scalars (real numbers) is denoted by $\Re$; the set of pairs (or triplets) of scalars, i.e., the two-dimensional plane (or three-dimensional space, respectively) is denoted by $\Re^2$ (or $\Re^3$, respectively).

Sets and the associated operations are easy to visualize in terms of **Venn diagrams**, as illustrated in Fig. 1.1.

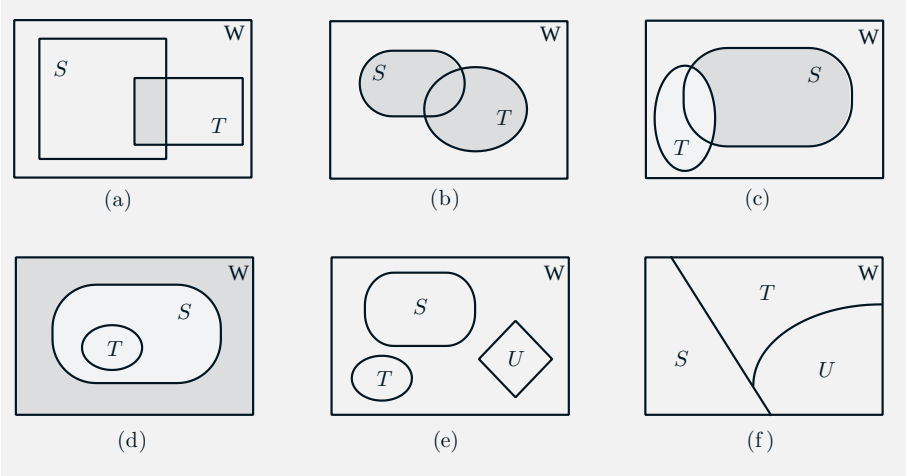

Figure 1.1: Examples of Venn diagrams. (a) The shaded region is $S \cap T$. (b) The shaded region is $S \cup T$. (c) The shaded region is $S \cap T^c$. (d) Here, $T \subset S$. The shaded region is the complement of S. (e) The sets S, T, and U are disjoint. (f) The sets S, T, and U form a partition of the set Ω.

The Algebra of Sets

Set operations have several properties, which are elementary consequences of the definitions. Some examples are:

$$\begin{aligned} S \cup T &= T \cup S, & S \cup (T \cup U) &= (S \cup T) \cup U, \\ S \cap (T \cup U) &= (S \cap T) \cup (S \cap U), & S \cup (T \cap U) &= (S \cup T) \cap (S \cup U), \\ (S^c)^c &= S, & S \cap S^c &= \emptyset, \\ S \cup \Omega &= \Omega, & S \cap \Omega &= S. \end{aligned}$$

Two particularly useful properties are given by **De Morgan's laws** which state that

$$\left(\bigcup_n S_n \right)^c = \bigcap_n S_n^c, \qquad \left(\bigcap_n S_n \right)^c = \bigcup_n S_n^c.$$

To establish the first law, suppose that $x \in (\cup_n S_n)^c$. Then, $x \notin \cup_n S_n$, which implies that for every n, we have $x \notin S_n$. Thus, x belongs to the complement

of every S_n, and $x \in \cap_n S_n^c$. This shows that $(\cup_n S_n)^c \subset \cap_n S_n^c$. The converse inclusion is established by reversing the above argument, and the first law follows. The argument for the second law is similar.

1.2 PROBABILISTIC MODELS

A probabilistic model is a mathematical description of an uncertain situation. It must be in accordance with a fundamental framework that we discuss in this section. Its two main ingredients are listed below and are visualized in Fig. 1.2.

Elements of a Probabilistic Model

- The **sample space** Ω, which is the set of all possible **outcomes** of an experiment.
- The **probability law**, which assigns to a set A of possible outcomes (also called an **event**) a nonnegative number $\mathbf{P}(A)$ (called the **probability** of A) that encodes our knowledge or belief about the collective "likelihood" of the elements of A. The probability law must satisfy certain properties to be introduced shortly.

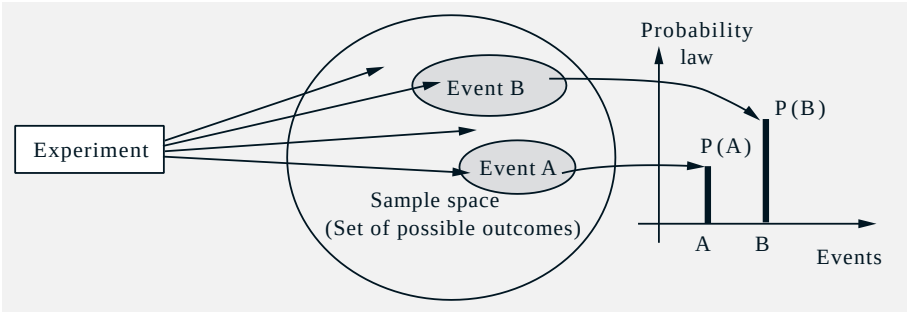

Figure 1.2: The main ingredients of a probabilistic model.

Sample Spaces and Events

Every probabilistic model involves an underlying process, called the **experiment**, that will produce exactly one out of several possible **outcomes**. The set of all possible outcomes is called the **sample space** of the experiment, and is denoted by Ω. A subset of the sample space, that is, a collection of possible

outcomes, is called an **event**.† There is no restriction on what constitutes an experiment. For example, it could be a single toss of a coin, or three tosses, or an infinite sequence of tosses. However, it is important to note that in our formulation of a probabilistic model, there is only one experiment. So, three tosses of a coin constitute a single experiment, rather than three experiments.

The sample space of an experiment may consist of a finite or an infinite number of possible outcomes. Finite sample spaces are conceptually and mathematically simpler. Still, sample spaces with an infinite number of elements are quite common. As an example, consider throwing a dart on a square target and viewing the point of impact as the outcome.

Choosing an Appropriate Sample Space

Regardless of their number, different elements of the sample space should be distinct and **mutually exclusive**, so that when the experiment is carried out there is a unique outcome. For example, the sample space associated with the roll of a die cannot contain "1 or 3" as a possible outcome and also "1 or 4" as another possible outcome. If it did, we would not be able to assign a unique outcome when the roll is a 1.

A given physical situation may be modeled in several different ways, depending on the kind of questions that we are interested in. Generally, the sample space chosen for a probabilistic model must be **collectively exhaustive**, in the sense that no matter what happens in the experiment, we always obtain an outcome that has been included in the sample space. In addition, the sample space should have enough detail to distinguish between all outcomes of interest to the modeler, while avoiding irrelevant details.

Example 1.1. Consider two alternative games, both involving ten successive coin tosses:

Game 1: We receive \$1 each time a head comes up.

Game 2: We receive \$1 for every coin toss, up to and including the first time a head comes up. Then, we receive \$2 for every coin toss, up to the second time a head comes up. More generally, the dollar amount per toss is doubled each time a head comes up.

† Any collection of possible outcomes, including the entire sample space Ω and its complement, the empty set $\emptyset$, may qualify as an event. Strictly speaking, however, some sets have to be excluded. In particular, when dealing with probabilistic models involving an uncountably infinite sample space, there are certain unusual subsets for which one cannot associate meaningful probabilities. This is an intricate technical issue, involving the mathematics of measure theory. Fortunately, such pathological subsets do not arise in the problems considered in this text or in practice, and the issue can be safely ignored.

In game 1, it is only the total number of heads in the ten-toss sequence that matters, while in game 2, the order of heads and tails is also important. Thus, in a probabilistic model for game 1, we can work with a sample space consisting of eleven possible outcomes, namely, $0, 1, \ldots, 10$. In game 2, a finer grain description of the experiment is called for, and it is more appropriate to let the sample space consist of every possible ten-long sequence of heads and tails.

Sequential Models

Many experiments have an inherently sequential character; for example, tossing a coin three times, observing the value of a stock on five successive days, or receiving eight successive digits at a communication receiver. It is then often useful to describe the experiment and the associated sample space by means of a **tree-based sequential description**, as in Fig. 1.3.

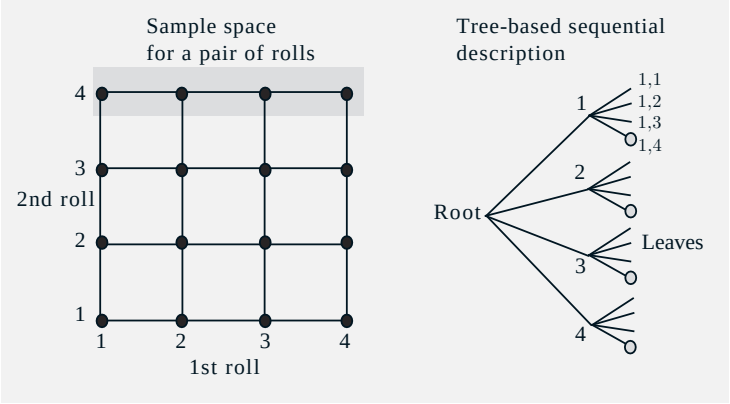

Figure 1.3: Two equivalent descriptions of the sample space of an experiment involving two rolls of a 4-sided die. The possible outcomes are all the ordered pairs of the form (i, j), where i is the result of the first roll, and j is the result of the second. These outcomes can be arranged in a 2-dimensional grid as in the figure on the left, or they can be described by the tree on the right, which reflects the sequential character of the experiment. Here, each possible outcome corresponds to a leaf of the tree and is associated with the unique path from the root to that leaf. The shaded area on the left is the event $\{(1, 4),\ (2, 4),\ (3, 4),\ (4, 4)\}$ that the result of the second roll is 4. That same event can be described by the set of leaves highlighted on the right. Note also that every node of the tree can be identified with an event, namely, the set of all leaves downstream from that node. For example, the node labeled by a 1 can be identified with the event $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$ that the result of the first roll is 1.

Probability Laws

Suppose we have settled on the sample space Ω associated with an experiment. To complete the probabilistic model, we must now introduce a **probability law**.

Intuitively, this specifies the "likelihood" of any outcome, or of any set of possible outcomes (an event, as we have called it earlier). More precisely, the probability law assigns to every event A, a number $\mathbf{P}(A)$, called the **probability** of A, satisfying the following axioms.

Probability Axioms

1. **(Nonnegativity)** $\mathbf{P}(A) \geq 0$, for every event A.
2. **(Additivity)** If A and B are two disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B).$$

   More generally, if the sample space has an infinite number of elements and $A_1, A_2, \ldots$ is a sequence of disjoint events, then the probability of their union satisfies

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots.$$

3. **(Normalization)** The probability of the entire sample space Ω is equal to 1, that is, $\mathbf{P}(\Omega) = 1$.

In order to visualize a probability law, consider a unit of mass which is "spread" over the sample space. Then, $\mathbf{P}(A)$ is simply the total mass that was assigned collectively to the elements of A. In terms of this analogy, the additivity axiom becomes quite intuitive: the total mass in a sequence of disjoint events is the sum of their individual masses.

A more concrete interpretation of probabilities is in terms of relative frequencies: a statement such as $\mathbf{P}(A) = 2/3$ often represents a belief that event A will occur in about two thirds out of a large number of repetitions of the experiment. Such an interpretation, though not always appropriate, can sometimes facilitate our intuitive understanding. It will be revisited in Chapter 5, in our study of limit theorems.

There are many natural properties of a probability law, which have not been included in the above axioms for the simple reason that they can be **derived** from them. For example, note that the normalization and additivity axioms imply that

$$1 = \mathbf{P}(\Omega) = \mathbf{P}(\Omega \cup \emptyset) = \mathbf{P}(\Omega) + \mathbf{P}(\emptyset) = 1 + \mathbf{P}(\emptyset),$$

and this shows that the probability of the empty event is 0:

$$\mathbf{P}(\emptyset) = 0.$$

As another example, consider three disjoint events A_1, A_2, and A_3. We can use the additivity axiom for two disjoint events repeatedly, to obtain

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= \mathbf{P}\big(A_1 \cup (A_2 \cup A_3)\big) \\ &= \mathbf{P}(A_1) + \mathbf{P}(A_2 \cup A_3) \\ &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3).\end{aligned}$$

Proceeding similarly, we obtain that the probability of the union of finitely many disjoint events is always equal to the sum of the probabilities of these events. More such properties will be considered shortly.

Discrete Models

Here is an illustration of how to construct a probability law starting from some common sense assumptions about a model.

Example 1.2. Consider an experiment involving a single coin toss. There are two possible outcomes, heads (H) and tails (T). The sample space is $\Omega = \{H, T\}$, and the events are

$$\{H, T\},\ \{H\},\ \{T\},\ \emptyset.$$

If the coin is fair, i.e., if we believe that heads and tails are "equally likely," we should assign equal probabilities to the two possible outcomes and specify that $\mathbf{P}(\{H\}) = \mathbf{P}(\{T\}) = 0.5$. The additivity axiom implies that

$$\mathbf{P}\big(\{H, T\}\big) = \mathbf{P}\big(\{H\}\big) + \mathbf{P}\big(\{T\}\big) = 1,$$

which is consistent with the normalization axiom. Thus, the probability law is given by

$$\mathbf{P}\big(\{H, T\}\big) = 1, \qquad \mathbf{P}\big(\{H\}\big) = 0.5, \qquad \mathbf{P}\big(\{T\}\big) = 0.5, \qquad \mathbf{P}(\emptyset) = 0,$$

and satisfies all three axioms.

Consider another experiment involving three coin tosses. The outcome will now be a 3-long string of heads or tails. The sample space is

$$\Omega = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}.$$

We assume that each possible outcome has the same probability of 1/8. Let us construct a probability law that satisfies the three axioms. Consider, as an example, the event

$$A = \{\text{exactly 2 heads occur}\} = \{HHT,\ HTH,\ THH\}.$$

Using additivity, the probability of A is the sum of the probabilities of its elements:

$$\begin{aligned}\mathbf{P}\big(\{HHT,\ HTH,\ THH\}\big) &= \mathbf{P}\big(\{HHT\}\big) + \mathbf{P}\big(\{HTH\}\big) + \mathbf{P}\big(\{THH\}\big) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{3}{8}.\end{aligned}$$

Similarly, the probability of any event is equal to 1/8 times the number of possible outcomes contained in the event. This defines a probability law that satisfies the three axioms.

By using the additivity axiom and by generalizing the reasoning in the preceding example, we reach the following conclusion.

Discrete Probability Law

If the sample space consists of a finite number of possible outcomes, then the probability law is specified by the probabilities of the events that consist of a single element. In particular, the probability of any event $\{s_1, s_2, \ldots, s_n\}$ is the sum of the probabilities of its elements:

$$\mathbf{P}\big(\{s_1, s_2, \ldots, s_n\}\big) = \mathbf{P}(s_1) + \mathbf{P}(s_2) + \cdots + \mathbf{P}(s_n).$$

Note that we are using here the simpler notation $\mathbf{P}(s_i)$ to denote the probability of the event $\{s_i\}$, instead of the more precise $\mathbf{P}(\{s_i\})$. This convention will be used throughout the remainder of the book.

In the special case where the probabilities $\mathbf{P}(s_1), \ldots, \mathbf{P}(s_n)$ are all the same (by necessity equal to $1/n$, in view of the normalization axiom), we obtain the following.

Discrete Uniform Probability Law

If the sample space consists of n possible outcomes which are equally likely (i.e., all single-element events have the same probability), then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{n}.$$

Let us provide a few more examples of sample spaces and probability laws.

Example 1.3. Consider the experiment of rolling a pair of 4-sided dice (cf. Fig. 1.4). We assume the dice are fair, and we interpret this assumption to mean that each of the sixteen possible outcomes [pairs (i, j), with $i, j = 1, 2, 3, 4$] has the same probability of 1/16. To calculate the probability of an event, we must count the number of elements of the event and divide by 16 (the total number of possible

outcomes). Here are some event probabilities calculated in this way:

$$\begin{aligned}
\mathbf{P}\big(\{\text{the sum of the rolls is even}\}\big) &= 8/16 = 1/2,\\
\mathbf{P}\big(\{\text{the sum of the rolls is odd}\}\big) &= 8/16 = 1/2,\\
\mathbf{P}\big(\{\text{the first roll is equal to the second}\}\big) &= 4/16 = 1/4,\\
\mathbf{P}\big(\{\text{the first roll is larger than the second}\}\big) &= 6/16 = 3/8,\\
\mathbf{P}\big(\{\text{at least one roll is equal to 4}\}\big) &= 7/16.
\end{aligned}$$

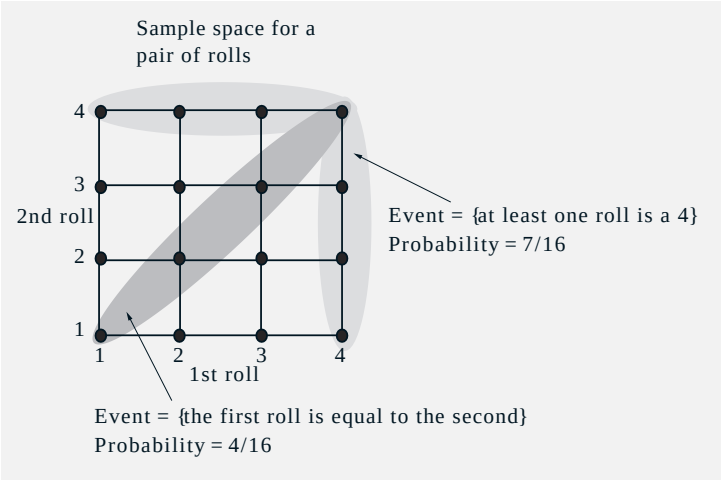

Figure 1.4: Various events in the experiment of rolling a pair of 4-sided dice, and their probabilities, calculated according to the discrete uniform law.

Continuous Models

Probabilistic models with continuous sample spaces differ from their discrete counterparts in that the probabilities of the single-element events may not be sufficient to characterize the probability law. This is illustrated in the following examples, which also indicate how to generalize the uniform probability law to the case of a continuous sample space.

Example 1.4. A wheel of fortune is continuously calibrated from 0 to 1, so the possible outcomes of an experiment consisting of a single spin are the numbers in the interval $\Omega = [0, 1]$. Assuming a fair wheel, it is appropriate to consider all outcomes equally likely, but what is the probability of the event consisting of a single element? It cannot be positive, because then, using the additivity axiom, it would follow that events with a sufficiently large number of elements would have

probability larger than 1. Therefore, the probability of any event that consists of a single element must be 0.

In this example, it makes sense to assign probability $b - a$ to any subinterval $[a, b]$ of $[0, 1]$, and to calculate the probability of a more complicated set by evaluating its "length."† This assignment satisfies the three probability axioms and qualifies as a legitimate probability law.

Example 1.5. Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

Let us use as sample space the unit square, whose elements are the possible pairs of delays for the two of them. Our interpretation of "equally likely" pairs of delays is to let the probability of a subset of Ω be equal to its area. This probability law satisfies the three probability axioms. The event that Romeo and Juliet will meet is the shaded region in Fig. 1.5, and its probability is calculated to be 7/16.

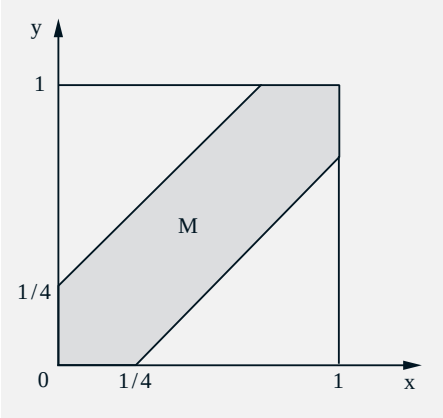

Figure 1.5: The event M that Romeo and Juliet will arrive within 15 minutes of each other (cf. Example 1.5) is

$$M = \Big\{(x, y) \;\Big|\; |x - y| \leq 1/4,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\Big\},$$

and is shaded in the figure. The area of M is 1 minus the area of the two unshaded triangles, or $1 - (3/4) \cdot (3/4) = 7/16$. Thus, the probability of meeting is 7/16.

† The "length" of a subset S of $[0, 1]$ is the integral $\int_S dt$, which is defined, for "nice" sets S, in the usual calculus sense. For unusual sets, this integral may not be well defined mathematically, but such issues belong to a more advanced treatment of the subject. Incidentally, the legitimacy of using length as a probability law hinges on the fact that the unit interval has an uncountably infinite number of elements. Indeed, if the unit interval had a countable number of elements, with each element having zero probability, the additivity axiom would imply that the whole interval has zero probability, which would contradict the normalization axiom.

Properties of Probability Laws

Probability laws have a number of properties, which can be deduced from the axioms. Some of them are summarized below.

Some Properties of Probability Laws

Consider a probability law, and let A, B, and C be events.

(a) If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$.

(b) $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$.

(c) $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$.

(d) $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$.

These properties, and other similar ones, can be visualized and verified graphically using Venn diagrams, as in Fig. 1.6. Note that property (c) can be generalized as follows:

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \mathbf{P}(A_i).$$

To see this, we apply property (c) to the sets A_1 and $A_2 \cup \cdots \cup A_n$, to obtain

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \mathbf{P}(A_1) + \mathbf{P}(A_2 \cup \cdots \cup A_n).$$

We also apply property (c) to the sets A_2 and $A_3 \cup \cdots \cup A_n$, to obtain

$$\mathbf{P}(A_2 \cup \cdots \cup A_n) \leq \mathbf{P}(A_2) + \mathbf{P}(A_3 \cup \cdots \cup A_n).$$

We continue similarly, and finally add.

Models and Reality

The framework of probability theory can be used to analyze uncertainty in a wide variety of physical contexts. Typically, this involves two distinct stages.

(a) In the first stage, we construct a probabilistic model by specifying a probability law on a suitably defined sample space. There are no hard rules to guide this step, other than the requirement that the probability law conform to the three axioms. Reasonable people may disagree on which model best represents reality. In many cases, one may even want to use a somewhat "incorrect" model, if it is simpler than the "correct" one or allows for tractable calculations. This is consistent with common practice in science

and engineering, where the choice of a model often involves a tradeoff between accuracy, simplicity, and tractability. Sometimes, a model is chosen on the basis of historical data or past outcomes of similar experiments, using statistical inference methods, which will be discussed in Chapters 8 and 9.

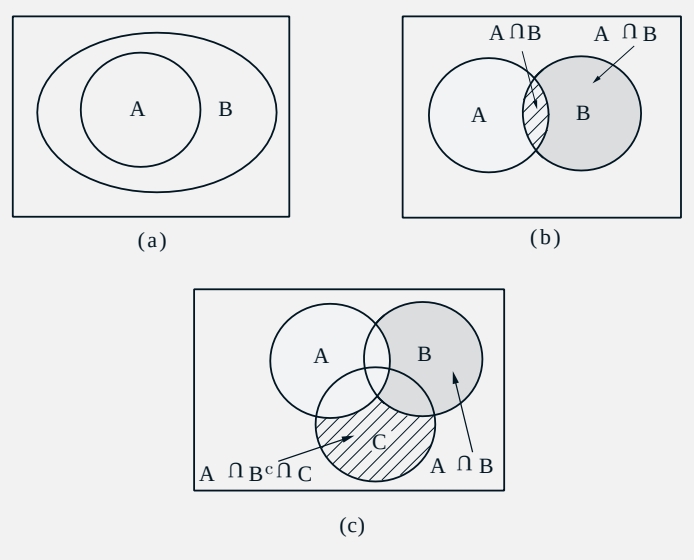

Figure 1.6: Visualization and verification of various properties of probability laws using Venn diagrams. If $A \subset B$, then B is the union of the two disjoint events A and $A^c \cap B$; see diagram (a). Therefore, by the additivity axiom, we have

$$\mathbf{P}(B) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) \geq \mathbf{P}(A),$$

where the inequality follows from the nonnegativity axiom, and verifies property (a).

From diagram (b), we can express the events $A \cup B$ and B as unions of disjoint events:

$$A \cup B = A \cup (A^c \cap B), \qquad B = (A \cap B) \cup (A^c \cap B).$$

Using the additivity axiom, we have

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B), \qquad \mathbf{P}(B) = \mathbf{P}(A \cap B) + \mathbf{P}(A^c \cap B).$$

Subtracting the second equality from the first and rearranging terms, we obtain $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$, verifying property (b). Using also the fact $\mathbf{P}(A \cap B) \geq 0$ (the nonnegativity axiom), we obtain $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$, verifying property (c).

From diagram (c), we see that the event $A \cup B \cup C$ can be expressed as a union of three disjoint events:

$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C),$$

so property (d) follows as a consequence of the additivity axiom.

(b) In the second stage, we work within a fully specified probabilistic model and derive the probabilities of certain events, or deduce some interesting properties. While the first stage entails the often open-ended task of connecting the real world with mathematics, the second one is tightly regulated by the rules of ordinary logic and the axioms of probability. Difficulties may arise in the latter if some required calculations are complex, or if a probability law is specified in an indirect fashion. Even so, there is no room for ambiguity: all conceivable questions have precise answers and it is only a matter of developing the skill to arrive at them.

Probability theory is full of "paradoxes" in which different calculation methods seem to give different answers to the same question. Invariably though, these apparent inconsistencies turn out to reflect poorly specified or ambiguous probabilistic models. An example, **Bertrand's paradox**, is shown in Fig. 1.7.

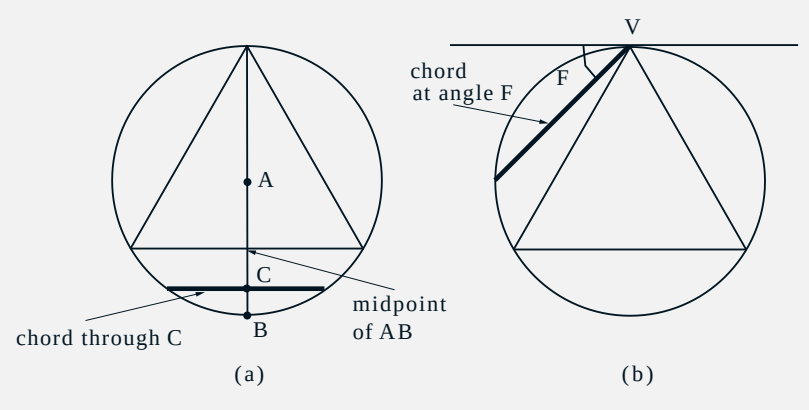

Figure 1.7: This example, presented by L. F. Bertrand in 1889, illustrates the need to specify unambiguously a probabilistic model. Consider a circle and an equilateral triangle inscribed in the circle. What is the probability that the length of a randomly chosen chord of the circle is greater than the side of the triangle? The answer here depends on the precise meaning of "randomly chosen." The two methods illustrated in parts (a) and (b) of the figure lead to contradictory results.

In (a), we take a radius of the circle, such as AB, and we choose a point C on that radius, with all points being equally likely. We then draw the chord through C that is orthogonal to AB. From elementary geometry, AB intersects the triangle at the midpoint of AB, so the probability that the length of the chord is greater than the side is $1/2$.

In (b), we take a point on the circle, such as the vertex V, we draw the tangent to the circle through V, and we draw a line through V that forms a random angle Φ with the tangent, with all angles being equally likely. We consider the chord obtained by the intersection of this line with the circle. From elementary geometry, the length of the chord is greater than the side of the triangle if Φ is between $\pi/3$ and $2\pi/3$. Since Φ takes values between 0 and π, the probability that the length of the chord is greater than the side is $1/3$.

A Brief History of Probability

- **B.C.E.** Games of chance were popular in ancient Greece and Rome, but no scientific development of the subject took place, possibly because the number system used by the Greeks did not facilitate algebraic calculations. The development of probability based on sound scientific analysis had to await the development of the modern arithmetic system by the Hindus and the Arabs in the second half of the first millennium, as well as the flood of scientific ideas generated by the Renaissance.

- **16th century.** Girolamo Cardano, a colorful and controversial Italian mathematician, publishes the first book describing correct methods for calculating probabilities in games of chance involving dice and cards.

- **17th century.** A correspondence between Fermat and Pascal touches upon several interesting probability questions and motivates further study in the field.

- **18th century.** Jacob Bernoulli studies repeated coin tossing and introduces the first law of large numbers, which lays a foundation for linking theoretical probability concepts and empirical fact. Several mathematicians, such as Daniel Bernoulli, Leibnitz, Bayes, and Lagrange, make important contributions to probability theory and its use in analyzing real-world phenomena. De Moivre introduces the normal distribution and proves the first form of the central limit theorem.

- **19th century.** Laplace publishes an influential book that establishes the importance of probability as a quantitative field and contains many original contributions, including a more general version of the central limit theorem. Legendre and Gauss apply probability to astronomical predictions, using the method of least squares, thus pointing the way to a vast range of applications. Poisson publishes an influential book with many original contributions, including the Poisson distribution. Chebyshev, and his students Markov and Lyapunov, study limit theorems and raise the standards of mathematical rigor in the field. Throughout this period, probability theory is largely viewed as a natural science, its primary goal being the explanation of physical phenomena. Consistently with this goal, probabilities are mainly interpreted as limits of relative frequencies in the context of repeatable experiments.

- **20th century.** Relative frequency is abandoned as the conceptual foundation of probability theory in favor of a now universally used axiomatic system, introduced by Kolmogorov. Similar to other branches of mathematics, the development of probability theory from the axioms relies only on logical correctness, regardless of its relevance to physical phenomena. Nonetheless, probability theory is used pervasively in science and engineering because of its ability to describe and interpret most types of uncertain phenomena in the real world.

1.3 CONDITIONAL PROBABILITY

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**. Here are some examples of situations we have in mind:

(a) In an experiment involving two successive rolls of a die, you are told that the sum of the two rolls is 9. How likely is it that the first roll was a 6?

(b) In a word guessing game, the first letter of the word is a "t". What is the likelihood that the second letter is an "h"?

(c) How likely is it that a person has a certain disease given that a medical test was negative?

(d) A spot shows up on a radar screen. How likely is it to correspond to an aircraft?

In more precise terms, given an experiment, a corresponding sample space, and a probability law, suppose that we know that the outcome is within some given event B. We wish to quantify the likelihood that the outcome also belongs to some other given event A. We thus seek to construct a new probability law that takes into account the available knowledge: a probability law that for any event A, specifies the **conditional probability of A given B**, denoted by $\mathbf{P}(A \,|\, B)$.

We would like the conditional probabilities $\mathbf{P}(A \,|\, B)$ of different events A to constitute a legitimate probability law, which satisfies the probability axioms. The conditional probabilities should also be consistent with our intuition in important special cases, e.g., when all possible outcomes of the experiment are equally likely. For example, suppose that all six possible outcomes of a fair die roll are equally likely. If we are told that the outcome is even, we are left with only three possible outcomes, namely, 2, 4, and 6. These three outcomes were equally likely to start with, and so they should remain equally likely given the additional knowledge that the outcome was even. Thus, it is reasonable to let

$$\mathbf{P}(\text{the outcome is } 6 \,|\, \text{the outcome is even}) = \frac{1}{3}.$$

This argument suggests that an appropriate definition of conditional probability when all outcomes are equally likely, is given by

$$\mathbf{P}(A \,|\, B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}.$$

Generalizing the argument, we introduce the following definition of conditional probability:

$$\mathbf{P}(A \,|\, B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

where we assume that $\mathbf{P}(B) > 0$; the conditional probability is undefined if the conditioning event has zero probability. In words, out of the total probability of the elements of B, $\mathbf{P}(A \mid B)$ is the fraction that is assigned to possible outcomes that also belong to A.

Conditional Probabilities Specify a Probability Law

For a fixed event B, it can be verified that the conditional probabilities $\mathbf{P}(A \mid B)$ form a legitimate probability law that satisfies the three axioms. Indeed, nonnegativity is clear. Furthermore,

$$\mathbf{P}(\Omega \mid B) = \frac{\mathbf{P}(\Omega \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B)}{\mathbf{P}(B)} = 1,$$

and the normalization axiom is also satisfied. To verify the additivity axiom, we write for any two disjoint events A_1 and A_2,

$$\begin{aligned}
\mathbf{P}(A_1 \cup A_2 \mid B) &= \frac{\mathbf{P}\big((A_1 \cup A_2) \cap B\big)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}((A_1 \cap B) \cup (A_2 \cap B))}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}(A_1 \cap B) + \mathbf{P}(A_2 \cap B)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}(A_1 \cap B)}{\mathbf{P}(B)} + \frac{\mathbf{P}(A_2 \cap B)}{\mathbf{P}(B)} \\
&= \mathbf{P}(A_1 \mid B) + \mathbf{P}(A_2 \mid B),
\end{aligned}$$

where for the third equality, we used the fact that $A_1 \cap B$ and $A_2 \cap B$ are disjoint sets, and the additivity axiom for the (unconditional) probability law. The argument for a countable collection of disjoint sets is similar.

Since conditional probabilities constitute a legitimate probability law, all general properties of probability laws remain valid. For example, a fact such as $\mathbf{P}(A \cup C) \leq \mathbf{P}(A) + \mathbf{P}(C)$ translates to the new fact

$$\mathbf{P}(A \cup C \mid B) \leq \mathbf{P}(A \mid B) + \mathbf{P}(C \mid B).$$

Let us also note that since we have $\mathbf{P}(B \mid B) = \mathbf{P}(B)/\mathbf{P}(B) = 1$, all of the conditional probability is concentrated on B. Thus, we might as well discard all possible outcomes outside B and treat the conditional probabilities as a probability law defined on the new universe B.

Let us summarize the conclusions reached so far.

Properties of Conditional Probability

- The conditional probability of an event A, given an event B with $\mathbf{P}(B) > 0$, is defined by

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

  and specifies a new (conditional) probability law on the same sample space Ω. In particular, all properties of probability laws remain valid for conditional probability laws.
- Conditional probabilities can also be viewed as a probability law on a new universe B, because all of the conditional probability is concentrated on B.
- If the possible outcomes are finitely many and equally likely, then

$$\mathbf{P}(A \mid B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}.$$

Example 1.6. We toss a fair coin three successive times. We wish to find the conditional probability $\mathbf{P}(A \mid B)$ when A and B are the events

$$A = \{\text{more heads than tails come up}\}, \qquad B = \{\text{1st toss is a head}\}.$$

The sample space consists of eight sequences,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

which we assume to be equally likely. The event B consists of the four elements HHH, HHT, HTH, HTT, so its probability is

$$\mathbf{P}(B) = \frac{4}{8}.$$

The event $A \cap B$ consists of the three elements HHH, HHT, HTH, so its probability is

$$\mathbf{P}(A \cap B) = \frac{3}{8}.$$

Thus, the conditional probability $\mathbf{P}(A \mid B)$ is

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Because all possible outcomes are equally likely here, we can also compute $\mathbf{P}(A \mid B)$ using a shortcut. We can bypass the calculation of $\mathbf{P}(B)$ and $\mathbf{P}(A \cap B)$, and simply

divide the number of elements shared by A and B (which is 3) with the number of elements of B (which is 4), to obtain the same result 3/4.

Example 1.7. A fair 4-sided die is rolled twice and we assume that all sixteen possible outcomes are equally likely. Let X and Y be the result of the 1st and the 2nd roll, respectively. We wish to determine the conditional probability $\mathbf{P}(A \,|\, B)$, where

$$A = \big\{\max(X,Y) = m\big\}, \qquad B = \big\{\min(X,Y) = 2\big\},$$

and m takes each of the values 1, 2, 3, 4.

As in the preceding example, we can first determine the probabilities $\mathbf{P}(A \cap B)$ and $\mathbf{P}(B)$ by counting the number of elements of $A \cap B$ and B, respectively, and dividing by 16. Alternatively, we can directly divide the number of elements of $A \cap B$ with the number of elements of B; see Fig. 1.8.

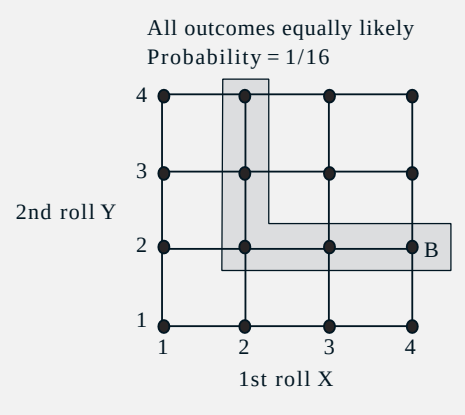

Figure 1.8: Sample space of an experiment involving two rolls of a 4-sided die. (cf. Example 1.7). The conditioning event $B = \{\min(X,Y) = 2\}$ consists of the 5-element shaded set. The set $A = \{\max(X,Y) = m\}$ shares with B two elements if $m = 3$ or $m = 4$, one element if $m = 2$, and no element if $m = 1$. Thus, we have

$$\mathbf{P}\Big(\{\max(X,Y) = m\} \;\Big|\; B\Big) = \begin{cases} 2/5, & \text{if } m = 3 \text{ or } m = 4, \\ 1/5, & \text{if } m = 2, \\ 0, & \text{if } m = 1. \end{cases}$$

Example 1.8. A conservative design team, call it C, and an innovative design team, call it N, are asked to separately design a new product within a month. From past experience we know that:

(a) The probability that team C is successful is 2/3.

(b) The probability that team N is successful is 1/2.

(c) The probability that at least one team is successful is 3/4.

Assuming that exactly one successful design is produced, what is the probability that it was designed by team N?

There are four possible outcomes here, corresponding to the four combinations of success and failure of the two teams:

SS: both succeed, FF: both fail,
SF: C succeeds, N fails, FS: C fails, N succeeds.

We were given that the probabilities of these outcomes satisfy

$$\mathbf{P}(SS) + \mathbf{P}(SF) = \frac{2}{3}, \quad \mathbf{P}(SS) + \mathbf{P}(FS) = \frac{1}{2}, \quad \mathbf{P}(SS) + \mathbf{P}(SF) + \mathbf{P}(FS) = \frac{3}{4}.$$

From these relations, together with the normalization equation

$$\mathbf{P}(SS) + \mathbf{P}(SF) + \mathbf{P}(FS) + \mathbf{P}(FF) = 1,$$

we can obtain the probabilities of individual outcomes:

$$\mathbf{P}(SS) = \frac{5}{12}, \qquad \mathbf{P}(SF) = \frac{1}{4}, \qquad \mathbf{P}(FS) = \frac{1}{12}, \qquad \mathbf{P}(FF) = \frac{1}{4}.$$

The desired conditional probability is

$$\mathbf{P}\big(FS \bigm| \{SF, FS\}\big) = \frac{\frac{1}{12}}{\frac{1}{4} + \frac{1}{12}} = \frac{1}{4}.$$

Using Conditional Probability for Modeling

When constructing probabilistic models for experiments that have a sequential character, it is often natural and convenient to first specify conditional probabilities and then use them to determine unconditional probabilities. The rule $\mathbf{P}(A \cap B) = \mathbf{P}(B)\mathbf{P}(A \mid B)$, which is a restatement of the definition of conditional probability, is often helpful in this process.

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

A sequential representation of the experiment is appropriate here, as shown in Fig. 1.9. Let A and B be the events

$$A = \{\text{an aircraft is present}\},$$
$$B = \{\text{the radar generates an alarm}\},$$

and consider also their complements

$$A^c = \{\text{an aircraft is not present}\},$$
$$B^c = \{\text{the radar does not generate an alarm}\}.$$

The given probabilities are recorded along the corresponding branches of the tree describing the sample space, as shown in Fig. 1.9. Each possible outcome corresponds to a leaf of the tree, and its probability is equal to the product of the probabilities associated with the branches in a path from the root to the corresponding leaf. The desired probabilities are

$$\mathbf{P}(\text{not present, false alarm}) = \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B \mid A^c) = 0.95 \cdot 0.10 = 0.095,$$
$$\mathbf{P}(\text{present, no detection}) = \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c \mid A) = 0.05 \cdot 0.01 = 0.0005.$$

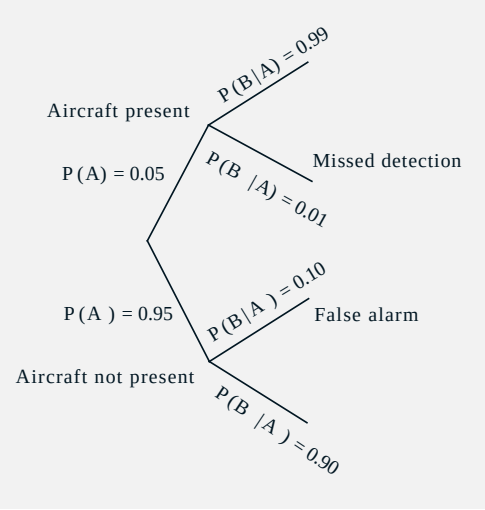

Figure 1.9: Sequential description of the experiment for the radar detection problem in Example 1.9.

Extending the preceding example, we have a general rule for calculating various probabilities in conjunction with a tree-based sequential description of an experiment. In particular:

(a) We set up the tree so that an event of interest is associated with a leaf. We view the occurrence of the event as a sequence of steps, namely, the traversals of the branches along the path from the root to the leaf.

(b) We record the conditional probabilities associated with the branches of the tree.

(c) We obtain the probability of a leaf by multiplying the probabilities recorded along the corresponding path of the tree.

In mathematical terms, we are dealing with an event A which occurs if and only if each one of several events $A_1, \ldots, A_n$ has occurred, i.e., $A = A_1 \cap A_2 \cap \cdots \cap A_n$. The occurrence of A is viewed as an occurrence of A_1, followed by the occurrence of A_2, then of A_3, etc., and it is visualized as a path with n branches, corresponding to the events $A_1, \ldots, A_n$. The probability of A is given by the following rule (see also Fig. 1.10).

Multiplication Rule

Assuming that all of the conditioning events have positive probability, we have

$$\mathbf{P}\big(\cap_{i=1}^{n} A_i\big) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2)\cdots\mathbf{P}\big(A_n \mid \cap_{i=1}^{n-1} A_i\big).$$

The multiplication rule can be verified by writing

$$\mathbf{P}\big(\cap_{i=1}^{n} A_i\big) = \mathbf{P}(A_1)\cdot\frac{\mathbf{P}(A_1\cap A_2)}{\mathbf{P}(A_1)}\cdot\frac{\mathbf{P}(A_1\cap A_2\cap A_3)}{\mathbf{P}(A_1\cap A_2)}\cdots\frac{\mathbf{P}\big(\cap_{i=1}^{n} A_i\big)}{\mathbf{P}\big(\cap_{i=1}^{n-1} A_i\big)},$$

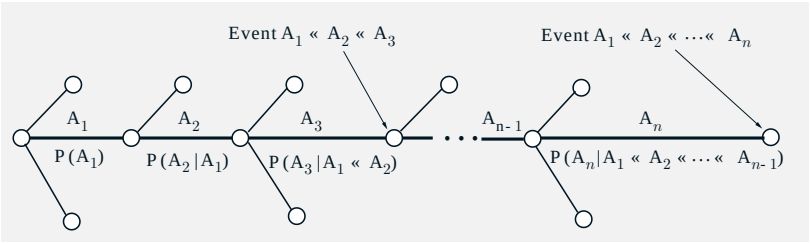

Figure 1.10: Visualization of the multiplication rule. The intersection event $A = A_1 \cap A_2 \cap \cdots \cap A_n$ is associated with a particular path on a tree that describes the experiment. We associate the branches of this path with the events $A_1, \ldots, A_n$, and we record next to the branches the corresponding conditional probabilities.

The final node of the path corresponds to the intersection event A, and its probability is obtained by multiplying the conditional probabilities recorded along the branches of the path

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\cdots\mathbf{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Note that any intermediate node along the path also corresponds to some intersection event and its probability is obtained by multiplying the corresponding conditional probabilities up to that node. For example, the event $A_1 \cap A_2 \cap A_3$ corresponds to the node shown in the figure, and its probability is

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2).$$

and by using the definition of conditional probability to rewrite the right-hand side above as

$$\mathbf{P}(A_1)\mathbf{P}(A_2 \,|\, A_1)\mathbf{P}(A_3 \,|\, A_1 \cap A_2) \cdots \mathbf{P}\big(A_n \,|\, \cap_{i=1}^{n-1} A_i\big).$$

For the case of just two events, A_1 and A_2, the multiplication rule is simply the definition of conditional probability.

Example 1.10. Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that none of the three cards is a heart. We assume that at each step, each one of the remaining cards is equally likely to be picked. By symmetry, this implies that every triplet of cards is equally likely to be drawn. A cumbersome approach, which we will not use, is to count the number of all card triplets that do not include a heart, and divide it with the number of all possible card triplets. Instead, we use a sequential description of the experiment in conjunction with the multiplication rule (cf. Fig. 1.11).

Define the events

$$A_i = \{\text{the } i\text{th card is not a heart}\}, \qquad i = 1, 2, 3.$$

We will calculate $\mathbf{P}(A_1 \cap A_2 \cap A_3)$, the probability that none of the three cards is a heart, using the multiplication rule

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \,|\, A_1)\mathbf{P}(A_3 \,|\, A_1 \cap A_2).$$

We have

$$\mathbf{P}(A_1) = \frac{39}{52},$$

since there are 39 cards that are not hearts in the 52-card deck. Given that the first card is not a heart, we are left with 51 cards, 38 of which are not hearts, and

$$\mathbf{P}(A_2 \,|\, A_1) = \frac{38}{51}.$$

Finally, given that the first two cards drawn are not hearts, there are 37 cards which are not hearts in the remaining 50-card deck, and

$$\mathbf{P}(A_3 \,|\, A_1 \cap A_2) = \frac{37}{50}.$$

These probabilities are recorded along the corresponding branches of the tree describing the sample space, as shown in Fig. 1.11. The desired probability is now obtained by multiplying the probabilities recorded along the corresponding path of the tree:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}.$$

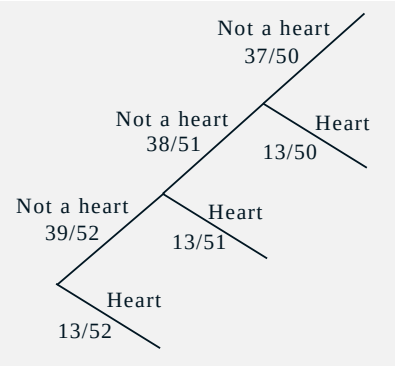

Figure 1.11: Sequential description of the experiment in the 3-card selection problem of Example 1.10.

Note that once the probabilities are recorded along the tree, the probability of several other events can be similarly calculated. For example,

$$\mathbf{P}(\text{1st is not a heart and 2nd is a heart}) = \frac{39}{52} \cdot \frac{13}{51},$$

$$\mathbf{P}(\text{1st and 2nd are not hearts, and 3rd is a heart}) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50}.$$

Example 1.11. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student? We interpret "randomly" to mean that given the assignment of some students to certain slots, any of the remaining students is equally likely to be assigned to any of the remaining slots. We then calculate the desired probability using the multiplication rule, based on the sequential description shown in Fig. 1.12. Let us denote the four graduate students by 1, 2, 3, 4, and consider the events

$$A_1 = \{\text{students 1 and 2 are in different groups}\},$$

$$A_2 = \{\text{students 1, 2, and 3 are in different groups}\},$$

$$A_3 = \{\text{students 1, 2, 3, and 4 are in different groups}\}.$$

We will calculate $\mathbf{P}(A_3)$ using the multiplication rule:

$$\mathbf{P}(A_3) = \mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \,|\, A_1)\mathbf{P}(A_3 \,|\, A_1 \cap A_2).$$

We have

$$\mathbf{P}(A_1) = \frac{12}{15},$$

since there are 12 student slots in groups other than the one of student 1, and there are 15 student slots overall, excluding student 1. Similarly,

$$\mathbf{P}(A_2 \,|\, A_1) = \frac{8}{14},$$

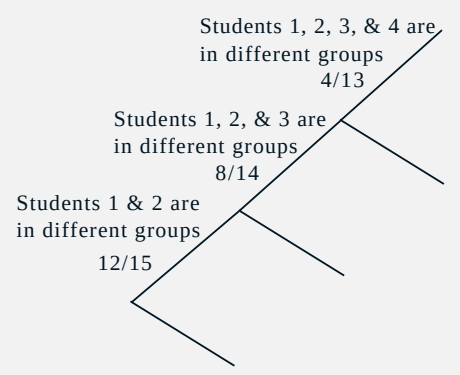

Figure 1.12: Sequential description of the experiment in the student problem of Example 1.11.

since there are 8 student slots in groups other than those of students 1 and 2, and there are 14 student slots, excluding students 1 and 2. Also,

$$\mathbf{P}(A_3 \,|\, A_1 \cap A_2) = \frac{4}{13},$$

since there are 4 student slots in groups other than those of students 1, 2, and 3, and there are 13 student slots, excluding students 1, 2, and 3. Thus, the desired probability is

$$\frac{12}{15} \cdot \frac{8}{14} \cdot \frac{4}{13},$$

and is obtained by multiplying the conditional probabilities along the corresponding path of the tree in Fig. 1.12.

Example 1.12. The Monty Hall Problem. This is a much discussed puzzle, based on an old American game show. You are told that a prize is equally likely to be found behind any one of three closed doors in front of you. You point to one of the doors. A friend opens for you one of the remaining two doors, after making sure that the prize is not behind it. At this point, you can stick to your initial choice, or switch to the other unopened door. You win the prize if it lies behind your final choice of a door. Consider the following strategies:

(a) Stick to your initial choice.

(b) Switch to the other unopened door.

(c) You first point to door 1. If door 2 is opened, you do not switch. If door 3 is opened, you switch.

Which is the best strategy? To answer the question, let us calculate the probability of winning under each of the three strategies.

Under the strategy of no switching, your initial choice will determine whether you win or not, and the probability of winning is 1/3. This is because the prize is equally likely to be behind each door.

Under the strategy of switching, if the prize is behind the initially chosen door (probability 1/3), you do not win. If it is not (probability 2/3), and given that

another door without a prize has been opened for you, you will get to the winning door once you switch. Thus, the probability of winning is now 2/3, so (b) is a better strategy than (a).

Consider now strategy (c). Under this strategy, there is insufficient information for determining the probability of winning. The answer depends on the way that your friend chooses which door to open. Let us consider two possibilities.

Suppose that if the prize is behind door 1, your friend always chooses to open door 2. (If the prize is behind door 2 or 3, your friend has no choice.) If the prize is behind door 1, your friend opens door 2, you do not switch, and you win. If the prize is behind door 2, your friend opens door 3, you switch, and you win. If the prize is behind door 3, your friend opens door 2, you do not switch, and you lose. Thus, the probability of winning is 2/3, so strategy (c) in this case is as good as strategy (b).

Suppose now that if the prize is behind door 1, your friend is equally likely to open either door 2 or 3. If the prize is behind door 1 (probability 1/3), and if your friend opens door 2 (probability 1/2), you do not switch and you win (probability 1/6). But if your friend opens door 3, you switch and you lose. If the prize is behind door 2, your friend opens door 3, you switch, and you win (probability 1/3). If the prize is behind door 3, your friend opens door 2, you do not switch and you lose. Thus, the probability of winning is $1/6 + 1/3 = 1/2$, so strategy (c) in this case is inferior to strategy (b).

1.4 TOTAL PROBABILITY THEOREM AND BAYES' RULE

In this section, we explore some applications of conditional probability. We start with the following theorem, which is often useful for computing the probabilities of various events, using a "divide-and-conquer" approach.

Total Probability Theorem

Let $A_1, \ldots, A_n$ be disjoint events that form a partition of the sample space (each possible outcome is included in exactly one of the events $A_1, \ldots, A_n$) and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B, we have

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B) \\ &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n).\end{aligned}$$

The theorem is visualized and proved in Fig. 1.13. Intuitively, we are partitioning the sample space into a number of scenarios (events) A_i. Then, the probability that B occurs is a weighted average of its conditional probability under each scenario, where each scenario is weighted according to its (unconditional) probability. One of the uses of the theorem is to compute the probability of various events B for which the conditional probabilities $\mathbf{P}(B \mid A_i)$ are known or

easy to derive. The key is to choose appropriately the partition $A_1, \ldots, A_n$, and this choice is often suggested by the problem structure. Here are some examples.

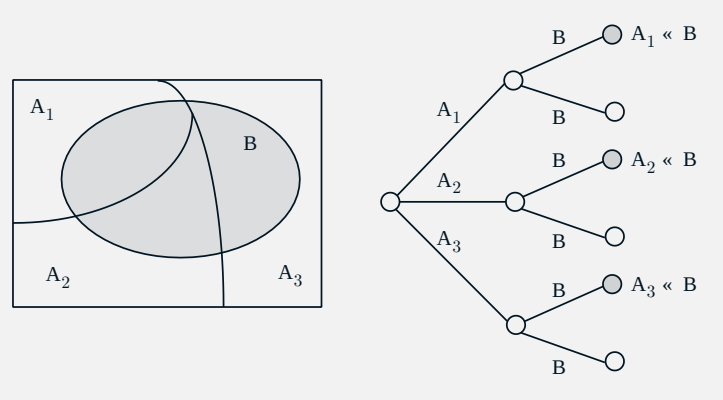

Figure 1.13: Visualization and verification of the total probability theorem. The events $A_1, \ldots, A_n$ form a partition of the sample space, so the event B can be decomposed into the disjoint union of its intersections $A_i \cap B$ with the sets A_i, i.e.,

$$B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B).$$

Using the additivity axiom, it follows that

$$\mathbf{P}(B) = \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B).$$

Since, by the definition of conditional probability, we have

$$\mathbf{P}(A_i \cap B) = \mathbf{P}(A_i)\mathbf{P}(B \mid A_i),$$

the preceding equality yields

$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n).$$

For an alternative view, consider an equivalent sequential model, as shown on the right. The probability of the leaf $A_i \cap B$ is the product $\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)$ of the probabilities along the path leading to that leaf. The event B consists of the three highlighted leaves and $\mathbf{P}(B)$ is obtained by adding their probabilities.

Example 1.13. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Let A_i be the event of playing with an opponent of type i. We have

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, let B be the event of winning. We have

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Thus, by the total probability theorem, the probability of winning is

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3) \\ &= 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 \\ &= 0.375.\end{aligned}$$

Example 1.14. You roll a fair four-sided die. If the result is 1 or 2, you roll once more but otherwise, you stop. What is the probability that the sum total of your rolls is at least 4?

Let A_i be the event that the result of first roll is i, and note that $\mathbf{P}(A_i) = 1/4$ for each i. Let B be the event that the sum total is at least 4. Given the event A_1, the sum total will be at least 4 if the second roll results in 3 or 4, which happens with probability 1/2. Similarly, given the event A_2, the sum total will be at least 4 if the second roll results in 2, 3, or 4, which happens with probability 3/4. Also, given the event A_3, you stop and the sum total remains below 4. Therefore,

$$\mathbf{P}(B \mid A_1) = \frac{1}{2}, \qquad \mathbf{P}(B \mid A_2) = \frac{3}{4}, \qquad \mathbf{P}(B \mid A_3) = 0, \qquad \mathbf{P}(B \mid A_4) = 1.$$

By the total probability theorem,

$$\mathbf{P}(B) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{9}{16}.$$

The total probability theorem can be applied repeatedly to calculate probabilities in experiments that have a sequential character, as shown in the following example.

Example 1.15. Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.4 (or 0.6, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

Let U_i and B_i be the events that Alice is up-to-date or behind, respectively, after i weeks. According to the total probability theorem, the desired probability $\mathbf{P}(U_3)$ is given by

$$\mathbf{P}(U_3) = \mathbf{P}(U_2)\mathbf{P}(U_3 \mid U_2) + \mathbf{P}(B_2)\mathbf{P}(U_3 \mid B_2) = \mathbf{P}(U_2) \cdot 0.8 + \mathbf{P}(B_2) \cdot 0.4.$$

The probabilities $\mathbf{P}(U_2)$ and $\mathbf{P}(B_2)$ can also be calculated using the total probability theorem:

$$\mathbf{P}(U_2) = \mathbf{P}(U_1)\mathbf{P}(U_2 \mid U_1) + \mathbf{P}(B_1)\mathbf{P}(U_2 \mid B_1) = \mathbf{P}(U_1) \cdot 0.8 + \mathbf{P}(B_1) \cdot 0.4,$$

$$\mathbf{P}(B_2) = \mathbf{P}(U_1)\mathbf{P}(B_2 \mid U_1) + \mathbf{P}(B_1)\mathbf{P}(B_2 \mid B_1) = \mathbf{P}(U_1) \cdot 0.2 + \mathbf{P}(B_1) \cdot 0.6.$$

Finally, since Alice starts her class up-to-date, we have

$$\mathbf{P}(U_1) = 0.8, \qquad \mathbf{P}(B_1) = 0.2.$$

We can now combine the preceding three equations to obtain

$$\mathbf{P}(U_2) = 0.8 \cdot 0.8 + 0.2 \cdot 0.4 = 0.72,$$

$$\mathbf{P}(B_2) = 0.8 \cdot 0.2 + 0.2 \cdot 0.6 = 0.28,$$

and by using the above probabilities in the formula for $\mathbf{P}(U_3)$:

$$\mathbf{P}(U_3) = 0.72 \cdot 0.8 + 0.28 \cdot 0.4 = 0.688.$$

Note that we could have calculated the desired probability $\mathbf{P}(U_3)$ by constructing a tree description of the experiment, then calculating the probability of every element of U_3 using the multiplication rule on the tree, and adding. However, there are cases where the calculation based on the total probability theorem is more convenient. For example, suppose we are interested in the probability $\mathbf{P}(U_{20})$ that Alice is up-to-date after 20 weeks. Calculating this probability using the multiplication rule is very cumbersome, because the tree representing the experiment is 20 stages deep and has 2^{20} leaves. On the other hand, with a computer, a sequential calculation using the total probability formulas

$$\mathbf{P}(U_{i+1}) = \mathbf{P}(U_i) \cdot 0.8 + \mathbf{P}(B_i) \cdot 0.4,$$

$$\mathbf{P}(B_{i+1}) = \mathbf{P}(U_i) \cdot 0.2 + \mathbf{P}(B_i) \cdot 0.6,$$

and the initial conditions $\mathbf{P}(U_1) = 0.8$, $\mathbf{P}(B_1) = 0.2$, is very simple.

Inference and Bayes' Rule

The total probability theorem is often used in conjunction with the following celebrated theorem, which relates conditional probabilities of the form $\mathbf{P}(A \mid B)$ to conditional probabilities of the form $\mathbf{P}(B \mid A)$, in which the order of the conditioning is reversed.

Bayes' Rule

Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B such that $\mathbf{P}(B) > 0$, we have

$$\begin{aligned}\mathbf{P}(A_i \mid B) &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)}.\end{aligned}$$

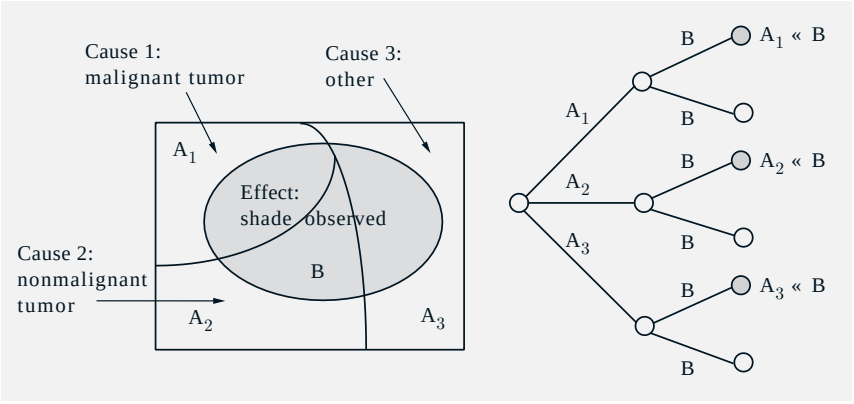

Figure 1.14: An example of the inference context that is implicit in Bayes' rule. We observe a shade in a person's X-ray (this is event B, the "effect") and we want to estimate the likelihood of three mutually exclusive and collectively exhaustive potential causes: cause 1 (event A_1) is that there is a malignant tumor, cause 2 (event A_2) is that there is a nonmalignant tumor, and cause 3 (event A_3) corresponds to reasons other than a tumor. We assume that we know the probabilities $\mathbf{P}(A_i)$ and $\mathbf{P}(B \mid A_i)$, $i = 1, 2, 3$. Given that we see a shade (event B occurs), Bayes' rule gives the posterior probabilities of the various causes as

$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)}, \quad i = 1, 2, 3.$$

For an alternative view, consider an equivalent sequential model, as shown on the right. The probability $\mathbf{P}(A_1 \mid B)$ of a malignant tumor is the probability of the first highlighted leaf, which is $\mathbf{P}(A_1 \cap B)$, divided by the total probability of the highlighted leaves, which is $\mathbf{P}(B)$.

To verify Bayes' rule, note that by the definition of conditional probability, we have

$$\mathbf{P}(A_i \cap B) = \mathbf{P}(A_i)\mathbf{P}(B \mid A_i) = \mathbf{P}(A_i \mid B)\mathbf{P}(B).$$

This yields the first equality. The second equality follows from the first by using the total probability theorem to rewrite $\mathbf{P}(B)$.

Bayes' rule is often used for **inference**. There are a number of "causes" that may result in a certain "effect." We observe the effect, and we wish to infer the cause. The events $A_1, \ldots, A_n$ are associated with the causes and the event B represents the effect. The probability $\mathbf{P}(B \mid A_i)$ that the effect will be observed when the cause A_i is present amounts to a probabilistic model of the cause-effect relation (cf. Fig. 1.14). Given that the effect B has been observed, we wish to evaluate the probability $\mathbf{P}(A_i \mid B)$ that the cause A_i is present. We refer to $\mathbf{P}(A_i \mid B)$ as the **posterior probability** of event A_i given the information, to be distinguished from $\mathbf{P}(A_i)$, which we call the **prior probability**.

Example 1.16. Let us return to the radar detection problem of Example 1.9 and Fig. 1.9. Let

$$A = \{\text{an aircraft is present}\},$$
$$B = \{\text{the radar generates an alarm}\}.$$

We are given that

$$\mathbf{P}(A) = 0.05, \qquad \mathbf{P}(B \mid A) = 0.99, \qquad \mathbf{P}(B \mid A^c) = 0.1.$$

Applying Bayes' rule, with $A_1 = A$ and $A_2 = A^c$, we obtain

$$\begin{aligned}
\mathbf{P}(\text{aircraft present} \mid \text{alarm}) &= \mathbf{P}(A \mid B) \\
&= \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(B)} \\
&= \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)} \\
&= \frac{0.05 \cdot 0.99}{0.05 \cdot 0.99 + 0.95 \cdot 0.1} \\
&\approx 0.3426.
\end{aligned}$$

Example 1.17. Let us return to the chess problem of Example 1.13. Here, A_i is the event of getting an opponent of type i, and

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, B is the event of winning, and

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Suppose that you win. What is the probability $\mathbf{P}(A_1 \mid B)$ that you had an opponent of type 1?

Using Bayes' rule, we have

$$\begin{aligned}
\mathbf{P}(A_1 \mid B) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)} \\
&= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} \\
&= 0.4.
\end{aligned}$$

Example 1.18. The False-Positive Puzzle. A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from

a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

If A is the event that the person has the disease, and B is the event that the test results are positive, the desired probability, $\mathbf{P}(A \mid B)$, is

$$\begin{aligned}\mathbf{P}(A \mid B) &= \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)} \\ &= \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} \\ &= 0.0187.\end{aligned}$$

Note that even though the test was assumed to be fairly accurate, a person who has tested positive is still very unlikely (less than 2%) to have the disease. According to *The Economist* (February 20th, 1999), 80% of those questioned at a leading American hospital substantially missed the correct answer to a question of this type; most of them thought that the probability that the person has the disease is 0.95!

1.5 INDEPENDENCE

We have introduced the conditional probability $\mathbf{P}(A \mid B)$ to capture the partial information that event B provides about event A. An interesting and important special case arises when the occurrence of B provides no such information and does not alter the probability that A has occurred, i.e.,

$$\mathbf{P}(A \mid B) = \mathbf{P}(A).$$

When the above equality holds, we say that A is **independent** of B. Note that by the definition $\mathbf{P}(A \mid B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$, this is equivalent to

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B).$$

We adopt this latter relation as the definition of independence because it can be used even when $\mathbf{P}(B) = 0$, in which case $\mathbf{P}(A \mid B)$ is undefined. The symmetry of this relation also implies that independence is a symmetric property; that is, if A is independent of B, then B is independent of A, and we can unambiguously say that A and B are **independent events**.

Independence is often easy to grasp intuitively. For example, if the occurrence of two events is governed by distinct and noninteracting physical processes, such events will turn out to be independent. On the other hand, independence is not easily visualized in terms of the sample space. A common first thought is that two events are independent if they are disjoint, but in fact the opposite is true: two disjoint events A and B with $\mathbf{P}(A) > 0$ and $\mathbf{P}(B) > 0$ are never independent, since their intersection $A \cap B$ is empty and has probability 0.

For example, an event A and its complement A^c are not independent [unless $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$], since knowledge that A has occurred provides precise information about whether A^c has occurred.

Example 1.19. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

(a) Are the events

$$A_i = \{\text{1st roll results in } i\}, \qquad B_j = \{\text{2nd roll results in } j\},$$

independent? We have

$$\mathbf{P}(A_i \cap B_j) = \mathbf{P}\big(\text{the outcome of the two rolls is } (i,j)\big) = \frac{1}{16},$$

$$\mathbf{P}(A_i) = \frac{\text{number of elements of } A_i}{\text{total number of possible outcomes}} = \frac{4}{16},$$

$$\mathbf{P}(B_j) = \frac{\text{number of elements of } B_j}{\text{total number of possible outcomes}} = \frac{4}{16}.$$

We observe that $\mathbf{P}(A_i \cap B_j) = \mathbf{P}(A_i)\mathbf{P}(B_j)$, and the independence of A_i and B_j is verified. Thus, our choice of the discrete uniform probability law implies the independence of the two rolls.

(b) Are the events

$$A = \{\text{1st roll is a 1}\}, \qquad B = \{\text{sum of the two rolls is a 5}\},$$

independent? The answer here is not quite obvious. We have

$$\mathbf{P}(A \cap B) = \mathbf{P}\big(\text{the result of the two rolls is (1,4)}\big) = \frac{1}{16},$$

and also

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{total number of possible outcomes}} = \frac{4}{16}.$$

The event B consists of the outcomes (1,4), (2,3), (3,2), and (4,1), and

$$\mathbf{P}(B) = \frac{\text{number of elements of } B}{\text{total number of possible outcomes}} = \frac{4}{16}.$$

Thus, we see that $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$, and the events A and B are independent.

(c) Are the events

$$A = \{\text{maximum of the two rolls is 2}\}, \ \ B = \{\text{minimum of the two rolls is 2}\},$$

independent? Intuitively, the answer is "no" because the minimum of the two rolls conveys some information about the maximum. For example, if the minimum is 2, the maximum cannot be 1. More precisely, to verify that A and B are not independent, we calculate

$$\mathbf{P}(A \cap B) = \mathbf{P}\big(\text{the result of the two rolls is } (2,2)\big) = \frac{1}{16},$$

and also

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{total number of possible outcomes}} = \frac{3}{16},$$

$$\mathbf{P}(B) = \frac{\text{number of elements of } B}{\text{total number of possible outcomes}} = \frac{5}{16}.$$

We have $\mathbf{P}(A)\mathbf{P}(B) = 15/(16)^2$, so that $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$, and A and B are not independent.

We finally note that, as mentioned earlier, if A and B are independent, the occurrence of B does not provide any new information on the probability of A occurring. It is then intuitive that the non-occurrence of B should also provide no information on the probability of A. Indeed, it can be verified that if A and B are independent, the same holds true for A and B^c (see the end-of-chapter problems).

Conditional Independence

We noted earlier that the conditional probabilities of events, conditioned on a particular event, form a legitimate probability law. We can thus talk about independence of various events with respect to this conditional law. In particular, given an event C, the events A and B are called **conditionally independent** if

$$\mathbf{P}(A \cap B \,|\, C) = \mathbf{P}(A \,|\, C)\mathbf{P}(B \,|\, C).$$

To derive an alternative characterization of conditional independence, we use the definition of the conditional probability and the multiplication rule, to write

$$\begin{aligned}\mathbf{P}(A \cap B \,|\, C) &= \frac{\mathbf{P}(A \cap B \cap C)}{\mathbf{P}(C)} \\ &= \frac{\mathbf{P}(C)\mathbf{P}(B \,|\, C)\mathbf{P}(A \,|\, B \cap C)}{\mathbf{P}(C)} \\ &= \mathbf{P}(B \,|\, C)\mathbf{P}(A \,|\, B \cap C).\end{aligned}$$

We now compare the preceding two expressions, and after eliminating the common factor $\mathbf{P}(B \,|\, C)$, assumed nonzero, we see that conditional independence is the same as the condition

$$\mathbf{P}(A \,|\, B \cap C) = \mathbf{P}(A \,|\, C).$$

In words, this relation states that if C is known to have occurred, the additional knowledge that B also occurred does not change the probability of A.

Interestingly, independence of two events A and B with respect to the unconditional probability law, does not imply conditional independence, and vice versa, as illustrated by the next two examples.

Example 1.20. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let

$$H_1 = \{\text{1st toss is a head}\},$$
$$H_2 = \{\text{2nd toss is a head}\},$$
$$D = \{\text{the two tosses have different results}\}.$$

The events H_1 and H_2 are (unconditionally) independent. But

$$\mathbf{P}(H_1 \,|\, D) = \frac{1}{2}, \qquad \mathbf{P}(H_2 \,|\, D) = \frac{1}{2}, \qquad \mathbf{P}(H_1 \cap H_2 \,|\, D) = 0,$$

so that $\mathbf{P}(H_1 \cap H_2 \,|\, D) \neq \mathbf{P}(H_1 \,|\, D)\mathbf{P}(H_2 \,|\, D)$, and H_1, H_2 are not conditionally independent.

This example can be generalized. For any probabilistic model, let A and B be independent events, and let C be an event such that $\mathbf{P}(C) > 0$, $\mathbf{P}(A \,|\, C) > 0$, and $\mathbf{P}(B \,|\, C) > 0$, while $A \cap B \cap C$ is empty. Then, A and B cannot be conditionally independent (given C) since $\mathbf{P}(A \cap B \,|\, C) = 0$ while $\mathbf{P}(A \,|\, C)\,\mathbf{P}(B \,|\, C) > 0$.

Example 1.21. There are two coins, a blue and a red one. We choose one of the two at random, each being chosen with probability 1/2, and proceed with two independent tosses. The coins are biased: with the blue coin, the probability of heads in any given toss is 0.99, whereas for the red coin it is 0.01.

Let B be the event that the blue coin was selected. Let also H_i be the event that the ith toss resulted in heads. Given the choice of a coin, the events H_1 and H_2 are independent, because of our assumption of independent tosses. Thus,

$$\mathbf{P}(H_1 \cap H_2 \,|\, B) = \mathbf{P}(H_1 \,|\, B)\mathbf{P}(H_2 \,|\, B) = 0.99 \cdot 0.99.$$

On the other hand, the events H_1 and H_2 are not independent. Intuitively, if we are told that the first toss resulted in heads, this leads us to suspect that the blue coin was selected, in which case, we expect the second toss to also result in heads. Mathematically, we use the total probability theorem to obtain

$$\mathbf{P}(H_1) = \mathbf{P}(B)\mathbf{P}(H_1 \,|\, B) + \mathbf{P}(B^c)\mathbf{P}(H_1 \,|\, B^c) = \frac{1}{2} \cdot 0.99 + \frac{1}{2} \cdot 0.01 = \frac{1}{2},$$

as should be expected from symmetry considerations. Similarly, we have $\mathbf{P}(H_2) = 1/2$. Now notice that

$$\begin{aligned}\mathbf{P}(H_1 \cap H_2) &= \mathbf{P}(B)\mathbf{P}(H_1 \cap H_2 \,|\, B) + \mathbf{P}(B^c)\mathbf{P}(H_1 \cap H_2 \,|\, B^c) \\ &= \frac{1}{2} \cdot 0.99 \cdot 0.99 + \frac{1}{2} \cdot 0.01 \cdot 0.01 \approx \frac{1}{2}.\end{aligned}$$

Thus, $\mathbf{P}(H_1 \cap H_2) \neq \mathbf{P}(H_1)\mathbf{P}(H_2)$, and the events H_1 and H_2 are dependent, even though they are conditionally independent given B.

We now summarize.

Independence

- Two events A and B are said to be **independent** if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B).$$

  If in addition, $\mathbf{P}(B) > 0$, independence is equivalent to the condition

$$\mathbf{P}(A \,|\, B) = \mathbf{P}(A).$$

- If A and B are independent, so are A and B^c.
- Two events A and B are said to be **conditionally independent**, given another event C with $\mathbf{P}(C) > 0$, if

$$\mathbf{P}(A \cap B \,|\, C) = \mathbf{P}(A \,|\, C)\mathbf{P}(B \,|\, C).$$

  If in addition, $\mathbf{P}(B \cap C) > 0$, conditional independence is equivalent to the condition

$$\mathbf{P}(A \,|\, B \cap C) = \mathbf{P}(A \,|\, C).$$

- Independence does not imply conditional independence, and vice versa.

Independence of a Collection of Events

The definition of independence can be extended to multiple events.

Definition of Independence of Several Events

We say that the events $A_1, A_2, \ldots, A_n$ are **independent** if

$$\mathbf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbf{P}(A_i), \qquad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}.$$

For the case of three events, A_1, A_2, and A_3, independence amounts to satisfying the four conditions

$$\begin{aligned}\mathbf{P}(A_1 \cap A_2) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2),\\ \mathbf{P}(A_1 \cap A_3) &= \mathbf{P}(A_1)\,\mathbf{P}(A_3),\\ \mathbf{P}(A_2 \cap A_3) &= \mathbf{P}(A_2)\,\mathbf{P}(A_3),\\ \mathbf{P}(A_1 \cap A_2 \cap A_3) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3).\end{aligned}$$

The first three conditions simply assert that any two events are independent, a property known as **pairwise independence**. But the fourth condition is also important and does not follow from the first three. Conversely, the fourth condition does not imply the first three; see the two examples that follow.

Example 1.22. Pairwise Independence does not Imply Independence. Consider two independent fair coin tosses, and the following events:

$$\begin{aligned}H_1 &= \{\text{1st toss is a head}\},\\ H_2 &= \{\text{2nd toss is a head}\},\\ D &= \{\text{the two tosses have different results}\}.\end{aligned}$$

The events H_1 and H_2 are independent, by definition. To see that H_1 and D are independent, we note that

$$\mathbf{P}(D \,|\, H_1) = \frac{\mathbf{P}(H_1 \cap D)}{\mathbf{P}(H_1)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathbf{P}(D).$$

Similarly, H_2 and D are independent. On the other hand, we have

$$\mathbf{P}(H_1 \cap H_2 \cap D) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{P}(H_1)\mathbf{P}(H_2)\mathbf{P}(D),$$

and these three events are not independent.

Example 1.23. The Equality $\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3)$ is not Enough for Independence. Consider two independent rolls of a fair six-sided die, and the following events:

$$\begin{aligned}A &= \{\text{1st roll is 1, 2, or 3}\},\\ B &= \{\text{1st roll is 3, 4, or 5}\},\\ C &= \{\text{the sum of the two rolls is 9}\}.\end{aligned}$$

We have

$$\mathbf{P}(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbf{P}(A)\mathbf{P}(B),$$

$$\mathbf{P}(A \cap C) = \frac{1}{36} \neq \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(A)\mathbf{P}(C),$$

$$\mathbf{P}(B \cap C) = \frac{1}{12} \neq \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(B)\mathbf{P}(C).$$

Thus the three events A, B, and C are not independent, and indeed no two of these events are independent. On the other hand, we have

$$\mathbf{P}(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{36} = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

The intuition behind the independence of a collection of events is analogous to the case of two events. Independence means that the occurrence or non-occurrence of **any number** of the events from that collection carries no information on the remaining events or their complements. For example, if the events A_1, A_2, A_3, A_4 are independent, one obtains relations such as

$$\mathbf{P}(A_1 \cup A_2 \,|\, A_3 \cap A_4) = \mathbf{P}(A_1 \cup A_2)$$

or

$$\mathbf{P}(A_1 \cup A_2^c \,|\, A_3^c \cap A_4) = \mathbf{P}(A_1 \cup A_2^c);$$

see the end-of-chapter problems.

Reliability

In probabilistic models of complex systems involving several components, it is often convenient to assume that the behaviors of the components are uncoupled (independent). This typically simplifies the calculations and the analysis, as illustrated in the following example.

Example 1.24. Network Connectivity. A computer network connects two nodes A and B through intermediate nodes C, D, E, F, as shown in Fig. 1.15(a). For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?

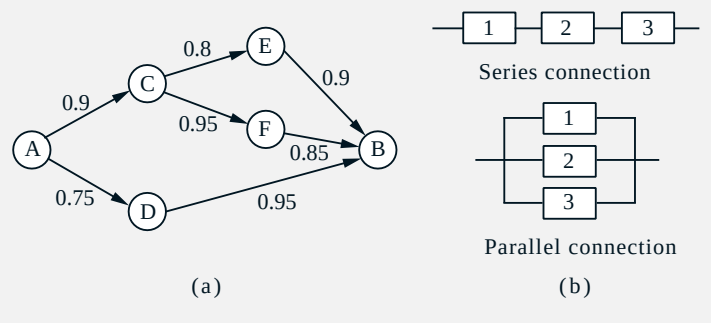

Figure 1.15: (a) Network for Example 1.24. The number next to each link indicates the probability that the link is up. (b) Series and parallel connections of three components in a reliability problem.

This is a typical problem of assessing the reliability of a system consisting of components that can fail independently. Such a system can often be divided into subsystems, where each subsystem consists in turn of several components that are connected either in **series** or in **parallel**; see Fig. 1.15(b).

Let a subsystem consist of components $1, 2, \ldots, m$, and let p_i be the probability that component i is up ("succeeds"). Then, a series subsystem succeeds if **all** of its components are up, so its probability of success is the product of the probabilities of success of the corresponding components, i.e.,

$$\mathbf{P}(\text{series subsystem succeeds}) = p_1 p_2 \cdots p_m.$$

A parallel subsystem succeeds if **any one** of its components succeeds, so its probability of failure is the product of the probabilities of failure of the corresponding components, i.e.,

$$\begin{aligned}\mathbf{P}(\text{parallel subsystem succeeds}) &= 1 - \mathbf{P}(\text{parallel subsystem fails}) \\ &= 1 - (1-p_1)(1-p_2)\cdots(1-p_m).\end{aligned}$$

Returning now to the network of Fig. 1.15(a), we can calculate the probability of success (a path from A to B is available) sequentially, using the preceding formulas, and starting from the end. Let us use the notation $X \to Y$ to denote the event that there is a (possibly indirect) connection from node X to node Y. Then,

$$\begin{aligned}\mathbf{P}(C \to B) &= 1 - \big(1 - \mathbf{P}(C \to E \text{ and } E \to B)\big)\big(1 - \mathbf{P}(C \to F \text{ and } F \to B)\big) \\ &= 1 - (1 - p_{CE}p_{EB})(1 - p_{CF}p_{FB}) \\ &= 1 - (1 - 0.8 \cdot 0.9)(1 - 0.95 \cdot 0.85) \\ &= 0.946,\end{aligned}$$

$$\mathbf{P}(A \to C \text{ and } C \to B) = \mathbf{P}(A \to C)\mathbf{P}(C \to B) = 0.9 \cdot 0.946 = 0.851,$$

$$\mathbf{P}(A \to D \text{ and } D \to B) = \mathbf{P}(A \to D)\mathbf{P}(D \to B) = 0.75 \cdot 0.95 = 0.712,$$

and finally we obtain the desired probability

$$\begin{aligned}\mathbf{P}(A \to B) &= 1 - \big(1 - \mathbf{P}(A \to C \text{ and } C \to B)\big)\big(1 - \mathbf{P}(A \to D \text{ and } D \to B)\big) \\ &= 1 - (1 - 0.851)(1 - 0.712) \\ &= 0.957.\end{aligned}$$

Independent Trials and the Binomial Probabilities

If an experiment involves a sequence of independent but identical stages, we say that we have a sequence of **independent trials**. In the special case where there are only two possible results at each stage, we say that we have a sequence of independent **Bernoulli trials**. The two possible results can be anything, e.g., "it rains" or "it doesn't rain," but we will often think in terms of coin tosses and refer to the two results as "heads" (H) and "tails" (T).

Consider an experiment that consists of n independent tosses of a coin, in which the probability of heads is p, where p is some number between 0 and 1. In this context, independence means that the events $A_1, A_2, \ldots, A_n$ are independent, where $A_i = \{i\text{th toss is a head}\}$.

We can visualize independent Bernoulli trials by means of a sequential description, as shown in Fig. 1.16 for the case where $n = 3$. The conditional probability of any toss being a head, conditioned on the results of any preceding tosses, is p because of independence. Thus, by multiplying the conditional probabilities along the corresponding path of the tree, we see that any particular outcome (3-long sequence of heads and tails) that involves k heads and $3 - k$ tails has probability $p^k(1-p)^{3-k}$. This formula extends to the case of a general number n of tosses. We obtain that the probability of any particular n-long sequence that contains k heads and $n - k$ tails is $p^k(1-p)^{n-k}$, for all k from 0 to n.

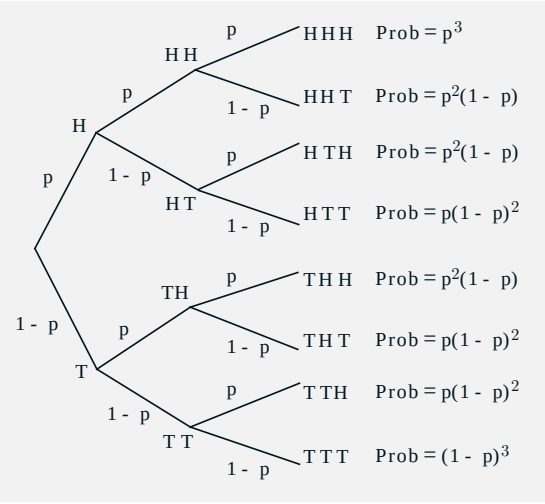

Figure 1.16: Sequential description of an experiment involving three independent tosses of a coin. Along the branches of the tree, we record the corresponding conditional probabilities, and by the multiplication rule, the probability of obtaining a particular 3-toss sequence is calculated by multiplying the probabilities recorded along the corresponding path of the tree.

Let us now consider the probability

$$p(k) = \mathbf{P}(k \text{ heads come up in an } n\text{-toss sequence}),$$

which will play an important role later. We showed above that the probability of any given sequence that contains k heads is $p^k(1-p)^{n-k}$, so we have

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where we use the notation

$$\binom{n}{k} = \text{number of distinct } n\text{-toss sequences that contain } k \text{ heads.}$$

The numbers $\binom{n}{k}$ (read as "n choose k") are known as the **binomial coefficients**, while the probabilities $p(k)$ are known as the **binomial probabilities**. Using a counting argument, to be given in Section 1.6, we can show that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \qquad k = 0, 1, \ldots, n,$$

where for any positive integer i we have

$$i! = 1 \cdot 2 \cdots (i-1) \cdot i,$$

and, by convention, $0! = 1$. An alternative verification is sketched in the end-of-chapter problems. Note that the binomial probabilities $p(k)$ must add to 1, thus showing the **binomial formula**

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

Example 1.25. Grade of Service. An internet service provider has installed c modems to serve the needs of a population of n dialup customers. It is estimated that at a given time, each customer will need a connection with probability p, independent of the others. What is the probability that there are more customers needing a connection than there are modems?

Here we are interested in the probability that more than c customers simultaneously need a connection. It is equal to

$$\sum_{k=c+1}^{n} p(k),$$

where

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

are the binomial probabilities. For instance, if $n = 100$, $p = 0.1$, and $c = 15$, the probability of interest turns out to be 0.0399.

This example is typical of problems of sizing a facility to serve the needs of a homogeneous population, consisting of independently acting customers. The problem is to select the facility size to guarantee a certain probability (sometimes called **grade of service**) that no user is left unserved.

1.6 COUNTING

The calculation of probabilities often involves counting the number of outcomes in various events. We have already seen two contexts where such counting arises.

(a) When the sample space Ω has a finite number of equally likely outcomes, so that the discrete uniform probability law applies. Then, the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega},$$

and involves counting the elements of A and of Ω.

(b) When we want to calculate the probability of an event A with a finite number of equally likely outcomes, each of which has an already known probability p. Then the probability of A is given by

$$\mathbf{P}(A) = p \cdot (\text{number of elements of } A),$$

and involves counting the number of elements of A. An example of this type is the calculation of the probability of k heads in n coin tosses (the binomial probabilities). We saw in the preceding section that the probability of each distinct sequence involving k heads is easily obtained, but the calculation of the number of all such sequences, to be presented shortly, requires some thought.

While counting is in principle straightforward, it is frequently challenging; the art of counting constitutes a large portion of the field of **combinatorics**. In this section, we present the basic principle of counting and apply it to a number of situations that are often encountered in probabilistic models.

The Counting Principle

The counting principle is based on a divide-and-conquer approach, whereby the counting is broken down into stages through the use of a tree. For example, consider an experiment that consists of two consecutive stages. The possible results at the first stage are $a_1, a_2, \ldots, a_m$; the possible results at the second stage are $b_1, b_2, \ldots, b_n$. Then, the possible results of the two-stage experiment are all possible **ordered** pairs (a_i, b_j), $i = 1, \ldots, m$, $j = 1, \ldots, n$. Note that the number of such ordered pairs is equal to mn. This observation can be generalized as follows (see also Fig. 1.17).

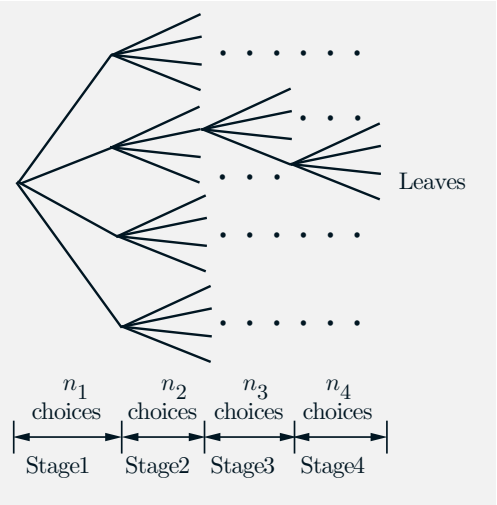

Figure 1.17: Illustration of the basic counting principle. The counting is carried out in r stages ($r = 4$ in the figure). The first stage has n_1 possible results. For every possible result at the first $i-1$ stages, there are n_i possible results at the ith stage. The number of leaves is $n_1 n_2 \cdots n_r$. This is the desired count.

The Counting Principle

Consider a process that consists of r stages. Suppose that:

(a) There are n_1 possible results at the first stage.

(b) For every possible result at the first stage, there are n_2 possible results at the second stage.

(c) More generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the ith stage.

Then, the total number of possible results of the r-stage process is

$$n_1 n_2 \cdots n_r.$$

Example 1.26. The Number of Telephone Numbers. A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How many distinct telephone numbers are there? We can visualize the choice of a sequence as a sequential process, where we select one digit at a time. We have a total of 7 stages, and a choice of one out of 10 elements at each stage, except for

the first stage where we only have 8 choices. Therefore, the answer is

$$8 \cdot \underbrace{10 \cdot 10 \cdots 10}_{6 \text{ times}} = 8 \cdot 10^6.$$

Example 1.27. The Number of Subsets of an n-Element Set. Consider an n-element set $\{s_1, s_2, \ldots, s_n\}$. How many subsets does it have (including itself and the empty set)? We can visualize the choice of a subset as a sequential process where we examine one element at a time and decide whether to include it in the set or not. We have a total of n stages, and a binary choice at each stage. Therefore the number of subsets is

$$\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n.$$

It should be noted that the Counting Principle remains valid even if each first-stage result leads to a different set of potential second-stage results, etc. The only requirement is that the number of possible second-stage results is constant, regardless of the first-stage result.

In what follows, we will focus primarily on two types of counting arguments that involve the selection of k objects out of a collection of n objects. If the order of selection matters, the selection is called a **permutation**, and otherwise, it is called a **combination**. We will then discuss a more general type of counting, involving a **partition** of a collection of n objects into multiple subsets.

k-permutations

We start with n distinct objects, and let k be some positive integer, with $k \leq n$. We wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence, i.e., the number of distinct k-object sequences. We can choose any of the n objects to be the first one. Having chosen the first, there are only $n-1$ possible choices for the second; given the choice of the first two, there only remain $n-2$ available objects for the third stage, etc. When we are ready to select the last (the kth) object, we have already chosen $k-1$ objects, which leaves us with $n-(k-1)$ choices for the last one. By the Counting Principle, the number of possible sequences, called **k-permutations**, is

$$\begin{aligned} n(n-1)\cdots(n-k+1) &= \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 2 \cdot 1}{(n-k)\cdots 2 \cdot 1} \\ &= \frac{n!}{(n-k)!}. \end{aligned}$$

In the special case where $k = n$, the number of possible sequences, simply called **permutations**, is

$$n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

(Let $k = n$ in the formula for the number of k-permutations, and recall the convention $0! = 1$.)

Example 1.28. Let us count the number of words that consist of four distinct letters. This is the problem of counting the number of 4-permutations of the 26 letters in the alphabet. The desired number is

$$\frac{n!}{(n-k)!} = \frac{26!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358,800.$$

The count for permutations can be combined with the Counting Principle to solve more complicated counting problems.

Example 1.29. You have n_1 classical music CDs, n_2 rock music CDs, and n_3 country music CDs. In how many different ways can you arrange them so that the CDs of the same type are contiguous?

We break down the problem in two stages, where we first select the order of the CD types, and then the order of the CDs of each type. There are 3! ordered sequences of the types of CDs (such as classical/rock/country, rock/country/classical, etc.), and there are $n_1!$ (or $n_2!$, or $n_3!$) permutations of the classical (or rock, or country, respectively) CDs. Thus for each of the 3! CD type sequences, there are $n_1!\, n_2!\, n_3!$ arrangements of CDs, and the desired total number is $3!\, n_1!\, n_2!\, n_3!$.

Suppose now that you offer to give k_i out of the n_i CDs of each type i to a friend, where $k_i < n_i$, $i = 1, 2, 3$. What is the number of all possible arrangements of the CDs that you are left with? The solution is similar, except that the number of $(n_i - k_i)$-permutations of CDs of type i replaces $n_i!$ in the estimate, so the number of possible arrangements is

$$3! \cdot \frac{n_1!}{k_1!} \cdot \frac{n_2!}{k_2!} \cdot \frac{n_3!}{k_3!}.$$

Combinations

There are n people and we are interested in forming a committee of k. How many different committees are possible? More abstractly, this is the same as the problem of counting the number of k-element subsets of a given n-element set. Notice that forming a combination is different than forming a k-permutation, because **in a combination there is no ordering of the selected elements**. For example, whereas the 2-permutations of the letters A, B, C, and D are

AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC,

the combinations of two out of these four letters are

AB, AC, AD, BC, BD, CD.

In the preceding example, the combinations are obtained from the permutations by grouping together "duplicates"; for example, AB and BA are not

viewed as distinct, and are both associated with the combination AB. This reasoning can be generalized: each combination is associated with $k!$ "duplicate" k-permutations, so the number $n!/(n-k)!$ of k-permutations is equal to the number of combinations times $k!$. Hence, the number of possible combinations, is equal to

$$\frac{n!}{k!\,(n-k)!}.$$

Let us now relate the above expression to the binomial coefficient, which was denoted by $\binom{n}{k}$ and was defined in the preceding section as the number of n-toss sequences with k heads. We note that specifying an n-toss sequence with k heads is the same as selecting k elements (those that correspond to heads) out of the n-element set of tosses, i.e., a combination of k out of n objects. Hence, the binomial coefficient is also given by the same formula and we have

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Example 1.30. The number of combinations of two out of the four letters A, B, C, and D is found by letting $n = 4$ and $k = 2$. It is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6,$$

consistent with the listing given earlier.

It is worth observing that counting arguments sometimes lead to formulas that are rather difficult to derive algebraically. One example is the **binomial formula**

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1,$$

discussed in Section 1.5. In the special case where $p = 1/2$, this formula becomes

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

and admits the following simple interpretation. Since $\binom{n}{k}$ is the number of k-element subsets of a given n-element subset, the sum over k of $\binom{n}{k}$ counts the number of subsets of all possible cardinalities. It is therefore equal to the number of all subsets of an n-element set, which is 2^n.

Example 1.31. We have a group of n persons. Consider clubs that consist of a special person from the group (the club leader) and a number (possibly zero) of

additional club members. Let us count the number of possible clubs of this type in two different ways, thereby obtaining an algebraic identity.

There are n choices for club leader. Once the leader is chosen, we are left with a set of $n-1$ available persons, and we are free to choose any of the 2^{n-1} subsets. Thus the number of possible clubs is $n2^{n-1}$.

Alternatively, for fixed k, we can form a k-person club by first selecting k out of the n available persons [there are $\binom{n}{k}$ choices]. We can then select one of the members to be the leader (there are k choices). By adding over all possible club sizes k, we obtain the number of possible clubs as $\sum_{k=1}^{n} k\binom{n}{k}$, thereby showing the identity

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Partitions

Recall that a combination is a choice of k elements out of an n-element set without regard to order. Thus, a combination can be viewed as a partition of the set in two: one part contains k elements and the other contains the remaining $n-k$. We now generalize by considering partitions into more than two subsets.

We are given an n-element set and nonnegative integers $n_1, n_2, \ldots, n_r$, whose sum is equal to n. We consider partitions of the set into r disjoint subsets, with the ith subset containing exactly n_i elements. Let us count in how many ways this can be done.

We form the subsets one at a time. We have $\binom{n}{n_1}$ ways of forming the first subset. Having formed the first subset, we are left with $n-n_1$ elements. We need to choose n_2 of them in order to form the second subset, and we have $\binom{n-n_1}{n_2}$ choices, etc. Using the Counting Principle for this r-stage process, the total number of choices is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r},$$

which is equal to

$$\frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{r-1})!}{(n-n_1-\cdots-n_{r-1}-n_r)!\,n_r!}.$$

We note that several terms cancel and we are left with

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

This is called the **multinomial coefficient** and is usually denoted by

$$\binom{n}{n_1, n_2, \ldots, n_r}.$$

Example 1.32. Anagrams. How many different words (letter sequences) can be obtained by rearranging the letters in the word TATTOO? There are six positions to be filled by the available letters. Each rearrangement corresponds to a partition of the set of the six positions into a group of size 3 (the positions that get the letter T), a group of size 1 (the position that gets the letter A), and a group of size 2 (the positions that get the letter O). Thus, the desired number is

$$\frac{6!}{1!\,2!\,3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 3} = 60.$$

It is instructive to derive this answer using an alternative argument. (This argument can also be used to rederive the multinomial coefficient formula; see the end-of-chapter problems.) Let us write TATTOO in the form $T_1AT_2T_3O_1O_2$ pretending for a moment that we are dealing with 6 distinguishable objects. These 6 objects can be rearranged in 6! different ways. However, any of the 3! possible permutations of T_1, T_2, and T_3, as well as any of the 2! possible permutations of O_1 and O_2, lead to the same word. Thus, when the subscripts are removed, there are only $6!/(3!\,2!)$ different words.

Example 1.33. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes a graduate student? This is the same as Example 1.11 in Section 1.3, but we will now obtain the answer using a counting argument.

We first determine the nature of the sample space. A typical outcome is a particular way of partitioning the 16 students into four groups of 4. We take the term "randomly" to mean that every possible partition is equally likely, so that the probability question can be reduced to one of counting.

According to our earlier discussion, there are

$$\binom{16}{4,4,4,4} = \frac{16!}{4!\,4!\,4!\,4!}$$

different partitions, and this is the size of the sample space.

Let us now focus on the event that each group contains a graduate student. Generating an outcome with this property can be accomplished in two stages:

(a) Take the four graduate students and distribute them to the four groups; there are four choices for the group of the first graduate student, three choices for the second, two for the third. Thus, there is a total of 4! choices for this stage.

(b) Take the remaining 12 undergraduate students and distribute them to the four groups (3 students in each). This can be done in

$$\binom{12}{3,3,3,3} = \frac{12!}{3!\,3!\,3!\,3!}$$

different ways.

By the Counting Principle, the event of interest can occur in

$$\frac{4!\,12!}{3!\,3!\,3!\,3!}$$

different ways. The probability of this event is

$$\frac{\dfrac{4!\,12!}{3!\,3!\,3!\,3!}}{\dfrac{16!}{4!\,4!\,4!\,4!}}.$$

After some cancellations, we find that this is equal to

$$\frac{12 \cdot 8 \cdot 4}{15 \cdot 14 \cdot 13},$$

consistent with the answer obtained in Example 1.11.

Here is a summary of all the counting results we have developed.

Summary of Counting Results

- **Permutations** of n objects: $n!$.
- k-**permutations** of n objects: $n!/(n-k)!$.
- **Combinations** of k out of n objects: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.
- **Partitions** of n objects into r groups, with the ith group having n_i objects:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\,n_2! \cdots n_r!}.$$

1.7 SUMMARY AND DISCUSSION

A probability problem can usually be broken down into a few basic steps:

(a) The description of the sample space, that is, the set of possible outcomes of a given experiment.

(b) The (possibly indirect) specification of the probability law (the probability of each event).

(c) The calculation of probabilities and conditional probabilities of various events of interest.

The probabilities of events must satisfy the nonnegativity, additivity, and normalization axioms. In the important special case where the set of possible outcomes is finite, one can just specify the probability of each outcome and obtain

the probability of any event by adding the probabilities of the elements of the event.

Given a probability law, we are often interested in conditional probabilities, which allow us to reason based on partial information about the outcome of the experiment. We can view conditional probabilities as probability laws of a special type, under which only outcomes contained in the conditioning event can have positive conditional probability. Conditional probabilities can be derived from the (unconditional) probability law using the definition $\mathbf{P}(A \mid B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$. However, the reverse process is often convenient, that is, first specify some conditional probabilities that are natural for the real situation that we wish to model, and then use them to derive the (unconditional) probability law.

We have illustrated through examples three methods for calculating probabilities:

(a) The **counting method**. This method applies to the case where the number of possible outcomes is finite, and all outcomes are equally likely. To calculate the probability of an event, we count the number of elements of the event and divide by the number of elements of the sample space.

(b) The **sequential method**. This method applies when the experiment has a sequential character, and suitable conditional probabilities are specified or calculated along the branches of the corresponding tree (perhaps using the counting method). The probabilities of various events are then obtained by multiplying conditional probabilities along the corresponding paths of the tree, using the multiplication rule.

(c) The **divide-and-conquer method**. Here, the probabilities $\mathbf{P}(B)$ of various events B are obtained from conditional probabilities $\mathbf{P}(B \mid A_i)$, where the A_i are suitable events that form a partition of the sample space and have known probabilities $\mathbf{P}(A_i)$. The probabilities $\mathbf{P}(B)$ are then obtained by using the total probability theorem.

Finally, we have focused on a few side topics that reinforce our main themes. We have discussed the use of Bayes' rule in inference, which is an important application context. We have also discussed some basic principles of counting and combinatorics, which are helpful in applying the counting method.

PROBLEMS

SECTION 1.1. Sets

Problem 1. Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate and compare the sets on both sides of De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c, \qquad \big(A \cap B\big)^c = A^c \cup B^c.$$

Problem 2. Let A and B be two sets.

(a) Show that

$$A^c = (A^c \cap B) \cup (A^c \cap B^c), \qquad B^c = (A \cap B^c) \cup (A^c \cap B^c).$$

(b) Show that

$$(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c).$$

(c) Consider rolling a fair six-sided die. Let A be the set of outcomes where the roll is an odd number. Let B be the set of outcomes where the roll is less than 4. Calculate the sets on both sides of the equality in part (b), and verify that the equality holds.

Problem 3.* Prove the identity

$$A \cup \big(\cap_{n=1}^{\infty} B_n\big) = \cap_{n=1}^{\infty}(A \cup B_n).$$

Solution. If x belongs to the set on the left, there are two possibilities. Either $x \in A$, in which case x belongs to all of the sets $A \cup B_n$, and therefore belongs to the set on the right. Alternatively, x belongs to all of the sets B_n in which case, it belongs to all of the sets $A \cup B_n$, and therefore again belongs to the set on the right.

Conversely, if x belongs to the set on the right, then it belongs to $A \cup B_n$ for all n. If x belongs to A, then it belongs to the set on the left. Otherwise, x must belong to every set B_n and again belongs to the set on the left.

Problem 4.* Cantor's diagonalization argument. Show that the unit interval $[0, 1]$ is uncountable, i.e., its elements cannot be arranged in a sequence.

Solution. Any number x in $[0, 1]$ can be represented in terms of its decimal expansion, e.g., $1/3 = 0.3333\cdots$. Note that most numbers have a unique decimal expansion, but there are a few exceptions. For example, $1/2$ can be represented as $0.5000\cdots$ or as $0.49999\cdots$. It can be shown that this is the only kind of exception, i.e., decimal expansions that end with an infinite string of zeroes or an infinite string of nines.

Suppose, to obtain a contradiction, that the elements of $[0,1]$ can be arranged in a sequence $x_1, x_2, x_3, \ldots$, so that every element of $[0,1]$ appears in the sequence. Consider the decimal expansion of x_n:

$$x_n = 0.a_n^1 a_n^2 a_n^3 \cdots,$$

where each digit a_n^i belongs to $\{0, 1, \ldots, 9\}$. Consider now a number y constructed as follows. The nth digit of y can be 1 or 2, and is chosen so that it is different from the nth digit of x_n. Note that y has a unique decimal expansion since it does not end with an infinite sequence of zeroes or nines. The number y differs from each x_n, since it has a different nth digit. Therefore, the sequence $x_1, x_2, \ldots$ does not exhaust the elements of $[0,1]$, contrary to what was assumed. The contradiction establishes that the set $[0,1]$ is uncountable.

SECTION 1.2. Probabilistic Models

Problem 5. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Problem 6. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

Problem 7. A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

Problem 8. You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

Problem 9. A partition of the sample space Ω is a collection of disjoint events $S_1, \ldots, S_n$ such that $\Omega = \cup_{i=1}^n S_i$.

(a) Show that for any event A, we have

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A \cap S_i).$$

(b) Use part (a) to show that for any events A, B, and C, we have

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) + \mathbf{P}\big(A \cap B^c \cap C^c\big) - \mathbf{P}(A \cap B \cap C).$$

Problem 10. Show the formula

$$\mathbf{P}\big((A \cap B^c) \cup (A^c \cap B)\big) = \mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A \cap B),$$

which gives the probability that exactly one of the events A and B will occur. [Compare with the formula $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$, which gives the probability that at least one of the events A and B will occur.]

Problem 11.* Bonferroni's inequality.

(a) Prove that for any two events A and B, we have

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1.$$

(b) Generalize to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots + \mathbf{P}(A_n) - (n-1).$$

Solution. We have $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$ and $\mathbf{P}(A \cup B) \leq 1$, which implies part (a). For part (b), we use De Morgan's law to obtain

$$\begin{aligned}
1 - \mathbf{P}(A_1 \cap \cdots \cap A_n) &= \mathbf{P}\big((A_1 \cap \cdots \cap A_n)^c\big) \\
&= \mathbf{P}(A_1^c \cup \cdots \cup A_n^c) \\
&\leq \mathbf{P}(A_1^c) + \cdots + \mathbf{P}(A_n^c) \\
&= \big(1 - \mathbf{P}(A_1)\big) + \cdots + \big(1 - \mathbf{P}(A_n)\big) \\
&= n - \mathbf{P}(A_1) - \cdots - \mathbf{P}(A_n).
\end{aligned}$$

Problem 12.* The inclusion-exclusion formula. Show the following generalizations of the formula

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

(a) Let A, B, and C be events. Then,

$$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap B) - \mathbf{P}(B \cap C) - \mathbf{P}(A \cap C) + \mathbf{P}(A \cap B \cap C).$$

(b) Let $A_1, A_2, \ldots, A_n$ be events. Let $S_1 = \{i \,|\, 1 \leq i \leq n\}$, $S_2 = \{(i_1, i_2) \,|\, 1 \leq i_1 < i_2 \leq n\}$, and more generally, let S_m be the set of all m-tuples $(i_1, \ldots, i_m)$ of indices that satisfy $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Then,

$$\begin{aligned}
\mathbf{P}\left(\cup_{k=1}^n A_k\right) = &\sum_{i \in S_1} \mathbf{P}(A_i) - \sum_{(i_1, i_2) \in S_2} \mathbf{P}(A_{i_1} \cap A_{i_2}) \\
&+ \sum_{(i_1, i_2, i_3) \in S_3} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n-1} \mathbf{P}\left(\cap_{k=1}^n A_k\right).
\end{aligned}$$

Solution. (a) We use the formulas $\mathbf{P}(X \cup Y) = \mathbf{P}(X) + \mathbf{P}(Y) - \mathbf{P}(X \cap Y)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. We have

$$\begin{aligned}
\mathbf{P}(A \cup B \cup C) &= \mathbf{P}(A \cup B) + \mathbf{P}(C) - \mathbf{P}\big((A \cup B) \cap C\big) \\
&= \mathbf{P}(A \cup B) + \mathbf{P}(C) - \mathbf{P}\big((A \cap C) \cup (B \cap C)\big) \\
&= \mathbf{P}(A \cup B) + \mathbf{P}(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) + \mathbf{P}(A \cap B \cap C) \\
&= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) + \mathbf{P}(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) \\
&\quad + \mathbf{P}(A \cap B \cap C) \\
&= \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap B) - \mathbf{P}(B \cap C) - \mathbf{P}(A \cap C) \\
&\quad + \mathbf{P}(A \cap B \cap C).
\end{aligned}$$

(b) Use induction and verify the main induction step by emulating the derivation of part (a). For a different approach, see the problems at the end of Chapter 2.

Problem 13.* Continuity property of probabilities.

(a) Let $A_1, A_2, \ldots$ be an infinite sequence of events, which is "monotonically increasing," meaning that $A_n \subset A_{n+1}$ for every n. Let $A = \cup_{n=1}^{\infty} A_n$. Show that $\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n)$. *Hint:* Express the event A as a union of countably many disjoint sets.

(b) Suppose now that the events are "monotonically decreasing," i.e., $A_{n+1} \subset A_n$ for every n. Let $A = \cap_{n=1}^{\infty} A_n$. Show that $\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n)$. *Hint:* Apply the result of part (a) to the complements of the events.

(c) Consider a probabilistic model whose sample space is the real line. Show that

$$\mathbf{P}\big([0,\infty)\big) = \lim_{n\to\infty} \mathbf{P}\big([0,n]\big), \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{P}\big([n,\infty)\big) = 0.$$

Solution. (a) Let $B_1 = A_1$ and, for $n \geq 2$, $B_n = A_n \cap A_{n-1}^c$. The events B_n are disjoint, and we have $\cup_{k=1}^{n} B_k = A_n$, and $\cup_{k=1}^{\infty} B_k = A$. We apply the additivity axiom to obtain

$$\mathbf{P}(A) = \sum_{k=1}^{\infty} \mathbf{P}(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{P}(B_k) = \lim_{n\to\infty} \mathbf{P}(\cup_{k=1}^{n} B_k) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

(b) Let $C_n = A_n^c$ and $C = A^c$. Since $A_{n+1} \subset A_n$, we obtain $C_n \subset C_{n+1}$, and the events C_n are increasing. Furthermore, $C = A^c = (\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c = \cup_{n=1}^{\infty} C_n$. Using the result from part (a) for the sequence C_n, we obtain

$$1 - \mathbf{P}(A) = \mathbf{P}(A^c) = \mathbf{P}(C) = \lim_{n\to\infty} \mathbf{P}(C_n) = \lim_{n\to\infty} \big(1 - \mathbf{P}(A_n)\big),$$

from which we conclude that $\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n)$.

(c) For the first equality, use the result from part (a) with $A_n = [0,n]$ and $A = [0,\infty)$. For the second, use the result from part (b) with $A_n = [n,\infty)$ and $A = \cap_{n=1}^{\infty} A_n = \emptyset$.

SECTION 1.3. Conditional Probability

Problem 14. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles are rolled.

(b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

(c) Find the probability that at least one die roll is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

Problem 15. A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one

of the tosses is a head. Is she right? Does it make a difference if the coin is fair or unfair? How can we generalize Alice's reasoning?

Problem 16. We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

Problem 17. A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives?

Problem 18. Let A and B be events. Show that $\mathbf{P}(A \cap B \,|\, B) = \mathbf{P}(A \,|\, B)$, assuming that $\mathbf{P}(B) > 0$.

SECTION 1.4. Total Probability Theorem and Bayes' Rule

Problem 19. Alice searches for her term paper in her filing cabinet, which has several drawers. She knows that she left her term paper in drawer j with probability $p_j > 0$. The drawers are so messy that even if she correctly guesses that the term paper is in drawer i, the probability that she finds it is only d_i. Alice searches in a particular drawer, say drawer i, but the search is unsuccessful. Conditioned on this event, show that the probability that her paper is in drawer j, is given by

$$\frac{p_j}{1 - p_i d_i}, \qquad \text{if } j \neq i, \qquad\qquad \frac{p_i(1 - d_i)}{1 - p_i d_i}, \qquad \text{if } j = i.$$

Problem 20. How an inferior player with a superior strategy can gain an advantage. Boris is about to play a two-game chess match with an opponent, and wants to find the strategy that maximizes his winning chances. Each game ends with either a win by one of the players, or a draw. If the score is tied at the end of the two games, the match goes into sudden-death mode, and the players continue to play until the first time one of them wins a game (and the match). Boris has two playing styles, *timid* and *bold*, and he can choose one of the two at will in each game, no matter what style he chose in previous games. With timid play, he draws with probability $p_d > 0$, and he loses with probability $1 - p_d$. With bold play, he wins with probability p_w, and he loses with probability $1 - p_w$. Boris will always play bold during sudden death, but may switch style between games 1 and 2.

(a) Find the probability that Boris wins the match for each of the following strategies:
   (i) Play bold in both games 1 and 2.
   (ii) Play timid in both games 1 and 2.
   (iii) Play timid whenever he is ahead in the score, and play bold otherwise.

(b) Assume that $p_w < 1/2$, so Boris is the worse player, regardless of the playing style he adopts. Show that with the strategy in (iii) above, and depending on the values of p_w and p_d, Boris may have a better than a 50-50 chance to win the match. How do you explain this advantage?

Problem 21. Two players take turns removing a ball from a jar that initially contains m white and n black balls. The first player to remove a white ball wins. Develop a

recursive formula that allows the convenient computation of the probability that the starting player wins.

Problem 22. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$.

Problem 23. We have two jars, each initially containing an equal number of balls. We perform four successive ball exchanges. In each exchange, we pick simultaneously and at random a ball from each jar and move it to the other jar. What is the probability that at the end of the four exchanges all the balls will be in the jar where they started?

Problem 24. The prisoner's dilemma. The release of two out of three prisoners has been announced, but their identity is kept secret. One of the prisoners considers asking a friendly guard to tell him who is the prisoner other than himself that will be released, but hesitates based on the following rationale: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released. What is wrong with this line of reasoning?

Problem 25. A two-envelopes puzzle. You are handed two envelopes, and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. A friend claims that the following strategy will increase above 1/2 your probability of ending up with the envelope with the larger amount: toss a coin repeatedly, let X be equal to 1/2 plus the number of tosses required to obtain heads for the first time, and switch if the amount in the envelope you selected is less than the value of X. Is your friend correct?

Problem 26. The paradox of induction. Consider a statement whose truth is unknown. If we see many examples that are compatible with it, we are tempted to view the statement as more probable. Such reasoning is often referred to as *inductive inference* (in a philosophical, rather than mathematical sense). Consider now the statement that "all cows are white." An equivalent statement is that "everything that is not white is not a cow." We then observe several black crows. Our observations are clearly compatible with the statement, but do they make the hypothesis "all cows are white" more likely?

To analyze such a situation, we consider a probabilistic model. Let us assume that there are two possible states of the world, which we model as complementary events:

$$A : \text{all cows are white,}$$

$$A^c : 50\% \text{ of all cows are white.}$$

Let p be the prior probability $\mathbf{P}(A)$ that all cows are white. We make an observation of a cow or a crow, with probability q and $1-q$, respectively, independent of whether

event A occurs or not. Assume that $0 < p < 1$, $0 < q < 1$, and that all crows are black.

(a) Given the event $B = \{\text{a black crow was observed}\}$, what is $\mathbf{P}(A \,|\, B)$?

(b) Given the event $C = \{\text{a white cow was observed}\}$, what is $\mathbf{P}(A \,|\, C)$?

Problem 27. Alice and Bob have $2n + 1$ coins, each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

Problem 28.* Conditional version of the total probability theorem. Let $C_1, \ldots, C_n$ be disjoint events that form a partition of the state space. Let also A and B be events such that $\mathbf{P}(B \cap C_i) > 0$ for all i. Show that

$$\mathbf{P}(A \,|\, B) = \sum_{i=1}^{n} \mathbf{P}(C_i \,|\, B)\mathbf{P}(A \,|\, B \cap C_i).$$

Solution. We have

$$\mathbf{P}(A \cap B) = \sum_{i=1}^{n} \mathbf{P}\big((A \cap B) \cap C_i\big),$$

and by using the multiplication rule,

$$\mathbf{P}\big(\big((A \cap B) \cap C_i\big) = \mathbf{P}(B)\mathbf{P}(C_i \,|\, B)\mathbf{P}(A \,|\, B \cap C_i).$$

Combining these two equations, dividing by $\mathbf{P}(B)$, and using the formula $\mathbf{P}(A \,|\, B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$, we obtain the desired result.

Problem 29.* Let A and B be events with $\mathbf{P}(A) > 0$ and $\mathbf{P}(B) > 0$. We say that an event B *suggests* an event A if $\mathbf{P}(A \,|\, B) > \mathbf{P}(A)$, and *does not suggest* event A if $\mathbf{P}(A \,|\, B) < \mathbf{P}(A)$.

(a) Show that B suggests A if and only if A suggests B.

(b) Assume that $\mathbf{P}(B^c) > 0$. Show that B suggests A if and only if B^c does not suggest A.

(c) We know that a treasure is located in one of two places, with probabilities β and $1-\beta$, respectively, where $0 < \beta < 1$. We search the first place and if the treasure is there, we find it with probability $p > 0$. Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place.

Solution. (a) We have $\mathbf{P}(A \,|\, B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$, so B suggests A if and only if $\mathbf{P}(A \cap B) > \mathbf{P}(A)\mathbf{P}(B)$, which is equivalent to A suggesting B, by symmetry.

(b) Since $\mathbf{P}(B) + \mathbf{P}(B^c) = 1$, we have

$$\mathbf{P}(B)\mathbf{P}(A) + \mathbf{P}(B^c)\mathbf{P}(A) = \mathbf{P}(A) = \mathbf{P}(B)\mathbf{P}(A \,|\, B) + \mathbf{P}(B^c)\mathbf{P}(A \,|\, B^c),$$

which implies that

$$\mathbf{P}(B^c)\big(\mathbf{P}(A) - \mathbf{P}(A \,|\, B^c)\big) = \mathbf{P}(B)\big(\mathbf{P}(A \,|\, B) - \mathbf{P}(A)\big).$$

Thus, $\mathbf{P}(A \mid B) > \mathbf{P}(A)$ (B suggests A) if and only if $\mathbf{P}(A) > \mathbf{P}(A \mid B^c)$ (B^c does not suggest A).

(c) Let A and B be the events

$$A = \{\text{the treasure is in the second place}\},$$
$$B = \{\text{we don't find the treasure in the first place}\}.$$

Using the total probability theorem, we have

$$\mathbf{P}(B) = \mathbf{P}(A^c)\mathbf{P}(B \mid A^c) + \mathbf{P}(A)\mathbf{P}(B \mid A) = \beta(1-p) + (1-\beta),$$

so

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{1-\beta}{\beta(1-p)+(1-\beta)} = \frac{1-\beta}{1-\beta p} > 1-\beta = \mathbf{P}(A).$$

It follows that event B suggests event A.

SECTION 1.5. Independence

Problem 30. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He knows that each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path, and if they agree, take that one, and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path?

Problem 31. Communication through a noisy channel. A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively, and is received incorrectly with probability ϵ_0 and ϵ_1, respectively (see Fig. 1.18). Errors in different symbol transmissions are independent.

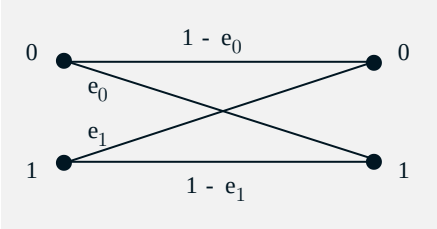

Figure 1.18: Error probabilities in a binary communication channel.

(a) What is the probability that the kth symbol is received correctly?

(b) What is the probability that the string of symbols 1011 is received correctly?

(c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is

transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that a 0 is correctly decoded?

(d) For what values of ϵ_0 is there an improvement in the probability of correct decoding of a 0 when the scheme of part (c) is used?

(e) Suppose that the scheme of part (c) is used. What is the probability that a symbol was 0 given that the received string is 101?

Problem 32. The king's sibling. The king has only one sibling. What is the probability that the sibling is male? Assume that every birth results in a boy with probability 1/2, independent of other births. Be careful to state any additional assumptions you have to make in order to arrive at an answer.

Problem 33. Using a biased coin to make an unbiased decision. Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased (though the bias is not known exactly). How can they use the biased coin to make a decision so that either option (opera or the movies) is equally likely to be chosen?

Problem 34. An electrical system consists of identical components, each of which is operational with probability p, independent of other components. The components are connected in three subsystems, as shown in Fig. 1.19. The system is operational if there is a path that starts at point A, ends at point B, and consists of operational components. What is the probability of this happening?

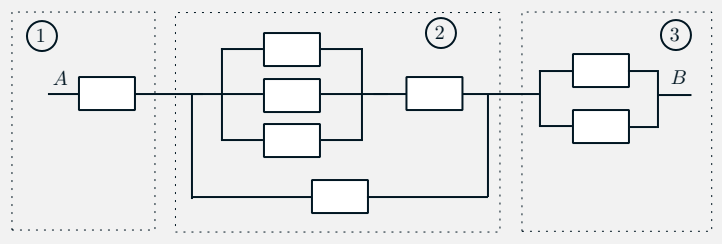

Figure 1.19: A system of identical components that consists of the three subsystems 1, 2, and 3. The system is operational if there is a path that starts at point A, ends at point B, and consists of operational components.

Problem 35. Reliability of a k-out-of-n system. A system consists of n identical components, each of which is operational with probability p, independent of other components. The system is operational if at least k out of the n components are operational. What is the probability that the system is operational?

Problem 36. A power utility can supply electricity to a city from n different power plants. Power plant i fails with probability p_i, independent of the others.

(a) Suppose that any one plant can produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

(b) Suppose that two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

Problem 37. A cellular phone system services a population of n_1 "voice users" (those who occasionally need a voice connection) and n_2 "data users" (those who occasionally need a data connection). We estimate that at a given time, each user will need to be connected to the system with probability p_1 (for voice users) or p_2 (for data users), independent of other users. The data rate for a voice user is r_1 bits/sec and for a data user is r_2 bits/sec. The cellular system has a total capacity of c bits/sec. What is the probability that more users want to use the system than the system can accommodate?

Problem 38. The problem of points. Telis and Wendy play a round of golf (18 holes) for a \$10 stake, and their probabilities of winning on any one hole are p and $1-p$, respectively, independent of their results in other holes. At the end of 10 holes, with the score 4 to 6 in favor of Wendy, Telis receives an urgent call and has to report back to work. They decide to split the stake in proportion to their probabilities of winning had they completed the round, as follows. If p_T and p_W are the conditional probabilities that Telis and Wendy, respectively, are ahead in the score after 18 holes given the 4-6 score after 10 holes, then Telis should get a fraction $p_T/(p_T+p_W)$ of the stake, and Wendy should get the remaining $p_W/(p_T+p_W)$. How much money should Telis get? *Note*: This is an example of the, so-called, problem of points, which played an important historical role in the development of probability theory. The problem was posed by Chevalier de Méré in the 17th century to Pascal, who introduced the idea that the stake of an interrupted game should be divided in proportion to the players' conditional probabilities of winning given the state of the game at the time of interruption. Pascal worked out some special cases and through a correspondence with Fermat, stimulated much thinking and several probability-related investigations.

Problem 39. A particular class has had a history of low attendance. The annoyed professor decides that she will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p_g if the weather is good, and with probability p_b if the weather is bad. Given the probability of bad weather on a given day, obtain an expression for the probability that the professor will teach her class on that day.

Problem 40. Consider a coin that comes up heads with probability p and tails with probability $1-p$. Let q_n be the probability that after n independent tosses, there have been an even number of heads. Derive a recursion that relates q_n to q_{n-1}, and solve this recursion to establish the formula

$$q_n = \big(1+(1-2p)^n\big)/2.$$

Problem 41. Consider a game show with an infinite pool of contestants, where at each round i, contestant i obtains a number by spinning a continuously calibrated wheel. The contestant with the smallest number thus far survives. Successive wheel spins are independent and we assume that there are no ties. Let N be the round at which contestant 1 is eliminated. For any positive integer n, find $\mathbf{P}(N=n)$.

Problem 42.* Gambler's ruin. A gambler makes a sequence of independent bets. In each bet, he wins \$1 with probability p, and loses \$1 with probability $1-p$. Initially, the gambler has \$$k$, and plays until he either accumulates \$$n$ or has no money left. What is the probability that the gambler will end up with \$$n$?

Solution. Let us denote by A the event that he ends up with \$$n$, and by F the event that he wins the first bet. Denote also by w_k the probability of event A, if he starts with \$$k$. We apply the total probability theorem to obtain

$$w_k = \mathbf{P}(A \,|\, F)\mathbf{P}(F) + \mathbf{P}(A \,|\, F^c)\mathbf{P}(F^c) = p\mathbf{P}(A \,|\, F) + q\mathbf{P}(A \,|\, F^c), \qquad 0 < k < n,$$

where $q = 1-p$. By the independence of past and future bets, having won the first bet is the same as if he were just starting now but with \$(k+1), so that $\mathbf{P}(A \,|\, F) = w_{k+1}$ and similarly $\mathbf{P}(A \,|\, F^c) = w_{k-1}$. Thus, we have $w_k = pw_{k+1} + qw_{k-1}$, which can be written as

$$w_{k+1} - w_k = r(w_k - w_{k-1}), \qquad 0 < k < n,$$

where $r = q/p$. We will solve for w_k in terms of p and q using iteration, and the boundary values $w_0 = 0$ and $w_n = 1$.

We have $w_{k+1} - w_k = r^k(w_1 - w_0)$, and since $w_0 = 0$,

$$w_{k+1} = w_k + r^k w_1 = w_{k-1} + r^{k-1} w_1 + r^k w_1 = w_1 + rw_1 + \cdots + r^k w_1.$$

The sum in the right-hand side can be calculated separately for the two cases where $r = 1$ (or $p = q$) and $r \neq 1$ (or $p \neq q$). We have

$$w_k = \begin{cases} \dfrac{1 - r^k}{1 - r} w_1, & \text{if } p \neq q, \\ kw_1, & \text{if } p = q. \end{cases}$$

Since $w_n = 1$, we can solve for w_1 and therefore for w_k:

$$w_1 = \begin{cases} \dfrac{1 - r}{1 - r^n}, & \text{if } p \neq q, \\ \dfrac{1}{n}, & \text{if } p = q, \end{cases}$$

so that

$$w_k = \begin{cases} \dfrac{1 - r^k}{1 - r^n}, & \text{if } p \neq q, \\ \dfrac{k}{n}, & \text{if } p = q. \end{cases}$$

Problem 43.* Let A and B be independent events. Use the definition of independence to prove the following:

(a) The events A and B^c are independent.

(b) The events A^c and B^c are independent.

Solution. (a) The event A is the union of the disjoint events $A \cap B^c$ and $A \cap B$. Using the additivity axiom and the independence of A and B, we obtain

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A \cap B^c).$$

It follows that

$$\mathbf{P}(A \cap B^c) = \mathbf{P}(A)\big(1 - \mathbf{P}(B)\big) = \mathbf{P}(A)\mathbf{P}(B^c),$$

so A and B^c are independent.

(b) Apply the result of part (a) twice: first on A and B, then on B^c and A.

Problem 44.* Let A, B, and C be independent events, with $\mathbf{P}(C) > 0$. Prove that A and B are conditionally independent given C.

Solution. We have

$$\begin{aligned}\mathbf{P}(A \cap B \,|\, C) &= \frac{\mathbf{P}(A \cap B \cap C)}{\mathbf{P}(C)} \\ &= \frac{\mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)}{\mathbf{P}(C)} \\ &= \mathbf{P}(A)\mathbf{P}(B) \\ &= \mathbf{P}(A \,|\, C)\mathbf{P}(B \,|\, C),\end{aligned}$$

so A and B are conditionally independent given C. In the preceding calculation, the first equality uses the definition of conditional probabilities; the second uses the assumed independence; the fourth uses the independence of A from C, and of B from C.

Problem 45.* Assume that the events A_1, A_2, A_3, A_4 are independent and that $\mathbf{P}(A_3 \cap A_4) > 0$. Show that

$$\mathbf{P}(A_1 \cup A_2 \,|\, A_3 \cap A_4) = \mathbf{P}(A_1 \cup A_2).$$

Solution. We have

$$\mathbf{P}(A_1 \,|\, A_3 \cap A_4) = \frac{\mathbf{P}(A_1 \cap A_3 \cap A_4)}{\mathbf{P}(A_3 \cap A_4)} = \frac{\mathbf{P}(A_1)\mathbf{P}(A_3)\mathbf{P}(A_4)}{\mathbf{P}(A_3)\mathbf{P}(A_4)} = \mathbf{P}(A_1).$$

We similarly obtain $\mathbf{P}(A_2 \,|\, A_3 \cap A_4) = \mathbf{P}(A_2)$ and $\mathbf{P}(A_1 \cap A_2 \,|\, A_3 \cap A_4) = \mathbf{P}(A_1 \cap A_2)$, and finally,

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \,|\, A_3 \cap A_4) &= \mathbf{P}(A_1 \,|\, A_3 \cap A_4) + \mathbf{P}(A_2 \,|\, A_3 \cap A_4) - \mathbf{P}(A_1 \cap A_2 \,|\, A_3 \cap A_4) \\ &= \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_1 \cap A_2) \\ &= \mathbf{P}(A_1 \cup A_2).\end{aligned}$$

Problem 46.* Laplace's rule of succession. Consider $m + 1$ boxes with the kth box containing k red balls and $m - k$ white balls, where k ranges from 0 to m. We choose a box at random (all boxes are equally likely) and then choose a ball at random from that box, n successive times (the ball drawn is replaced each time, and a new ball is selected independently). Suppose a red ball was drawn each of the n times. What is the probability that if we draw a ball one more time it will be red? Estimate this probability for large m.

Solution. We want to find the conditional probability $\mathbf{P}(E \,|\, R_n)$, where E is the event of a red ball drawn at time $n+1$, and R_n is the event of a red ball drawn each of the n preceding times. Intuitively, the consistent draw of a red ball indicates that a box with

a high percentage of red balls was chosen, so we expect that $\mathbf{P}(E \mid R_n)$ is closer to 1 than to 0. In fact, Laplace used this example to calculate the probability that the sun will rise tomorrow given that it has risen for the preceding 5,000 years. (It is not clear how serious Laplace was about this calculation, but the story is part of the folklore of probability theory.)

We have

$$\mathbf{P}(E \mid R_n) = \frac{\mathbf{P}(E \cap R_n)}{\mathbf{P}(R_n)},$$

and by using the total probability theorem, we obtain

$$\mathbf{P}(R_n) = \sum_{k=0}^{m} \mathbf{P}(k\text{th box chosen}) \left(\frac{k}{m}\right)^n = \frac{1}{m+1} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^n,$$

$$\mathbf{P}(E \cap R_n) = \mathbf{P}(R_{n+1}) = \frac{1}{m+1} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^{n+1}.$$

For large m, we can view $\mathbf{P}(R_n)$ as a piecewise constant approximation to an integral:

$$\mathbf{P}(R_n) = \frac{1}{m+1} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^n \approx \frac{1}{(m+1)m^n} \int_0^m x^n dx = \frac{1}{(m+1)m^n} \cdot \frac{m^{n+1}}{n+1} \approx \frac{1}{n+1}.$$

Similarly,

$$\mathbf{P}(E \cap R_n) = \mathbf{P}(R_{n+1}) \approx \frac{1}{n+2},$$

so that

$$\mathbf{P}(E \mid R_n) \approx \frac{n+1}{n+2}.$$

Thus, for large m, drawing a red ball one more time is almost certain when n is large.

Problem 47.* Binomial coefficient formula and the Pascal triangle.

(a) Use the definition of $\binom{n}{k}$ as the number of distinct n-toss sequences with k heads, to derive the recursion suggested by the so called Pascal triangle, given in Fig. 1.20.

(b) Use the recursion derived in part (a) and induction, to establish the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Solution. (a) Note that n-toss sequences that contain k heads (for $0 < k < n$) can be obtained in two ways:

(1) By starting with an $(n-1)$-toss sequence that contains k heads and adding a tail at the end. There are $\binom{n-1}{k}$ different sequences of this type.

(2) By starting with an $(n-1)$-toss sequence that contains $k-1$ heads and adding a head at the end. There are $\binom{n-1}{k-1}$ different sequences of this type.

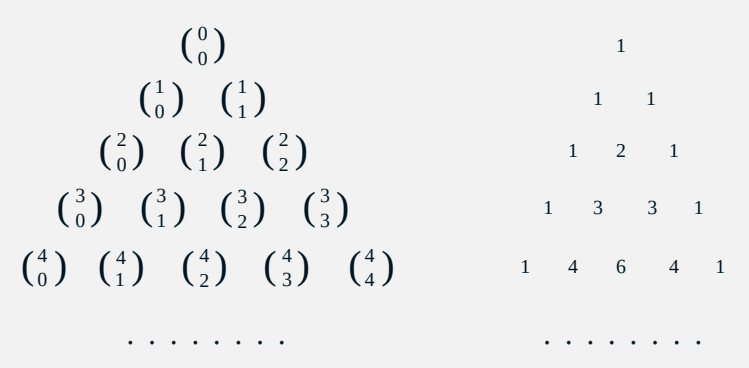

Figure 1.20: Sequential calculation method of the binomial coefficients using the Pascal triangle. Each term $\binom{n}{k}$ in the triangular array on the left is computed and placed in the triangular array on the right by adding its two neighbors in the row above it (except for the boundary terms with $k = 0$ or $k = n$, which are equal to 1).

Thus,

$$\binom{n}{k} = \begin{cases} \binom{n-1}{k-1} + \binom{n-1}{k}, & \text{if } k = 1, 2, \ldots, n-1, \\ 1, & \text{if } k = 0, n. \end{cases}$$

This is the formula corresponding to the Pascal triangle calculation, given in Fig. 1.20.

(b) We now use the recursion from part (a), to demonstrate the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

by induction on n. Indeed, we have from the definition $\binom{1}{0} = \binom{1}{1} = 1$, so for $n = 1$ the above formula is seen to hold as long as we use the convention $0! = 1$. If the formula holds for each index up to $n-1$, we have for $k = 1, 2, \ldots, n-1$,

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!} + \frac{(n-1)!}{k!\,(n-1-k)!} \\ &= \frac{k}{n} \cdot \frac{n!}{k!\,(n-k)!} + \frac{n-k}{n} \cdot \frac{n!}{k!\,(n-k)!} \\ &= \frac{n!}{k!\,(n-k)!}, \end{aligned}$$

and the induction is complete.

Problem 48.* The Borel-Cantelli lemma. Consider an infinite sequence of trials. The probability of success at the ith trial is some positive number p_i. Let N be the

event that there is no success, and let I be the event that there is an infinite number of successes.

(a) Assume that the trials are independent and that $\sum_{i=1}^{\infty} p_i = \infty$. Show that $\mathbf{P}(N) = 0$ and $\mathbf{P}(I) = 1$.

(b) Assume that $\sum_{i=1}^{\infty} p_i < \infty$. Show that $\mathbf{P}(I) = 0$.

Solution. (a) The event N is a subset of the event that there were no successes in the first n trials, so that

$$\mathbf{P}(N) \leq \prod_{i=1}^{n}(1 - p_i).$$

Taking logarithms,

$$\log \mathbf{P}(N) \leq \sum_{i=1}^{n} \log(1 - p_i) \leq \sum_{i=1}^{n}(-p_i).$$

Taking the limit as n tends to infinity, we obtain $\log \mathbf{P}(N) = -\infty$, or $\mathbf{P}(N) = 0$.

Let now L_n be the event that there is a finite number of successes and that the last success occurs at the nth trial. We use the already established result $\mathbf{P}(N) = 0$, and apply it to the sequence of trials after trial n, to obtain $\mathbf{P}(L_n) = 0$. The event I^c (finite number of successes) is the union of the disjoint events L_n, $n \geq 1$, and N, so that

$$\mathbf{P}(I^c) = \mathbf{P}(N) + \sum_{n=1}^{\infty} \mathbf{P}(L_n) = 0,$$

and $\mathbf{P}(I) = 1$.

(b) Let S_i be the event that the ith trial is a success. Fix some number n and for every $i > n$, let F_i be the event that the first success after time n occurs at time i. Note that $F_i \subset S_i$. Finally, let A_n be the event that there is at least one success after time n. Note that $I \subset A_n$, because an infinite number of successes implies that there are successes subsequent to time n. Furthermore, the event A_n is the union of the disjoint events F_i, $i > n$. Therefore,

$$\mathbf{P}(I) \leq \mathbf{P}(A_n) = \mathbf{P}\left(\bigcup_{i=n+1}^{\infty} F_i\right) = \sum_{i=n+1}^{\infty} \mathbf{P}(F_i) \leq \sum_{i=n+1}^{\infty} \mathbf{P}(S_i) = \sum_{i=n+1}^{\infty} p_i.$$

We take the limit of both sides as $n \to \infty$. Because of the assumption $\sum_{i=1}^{\infty} p_i < \infty$, the right-hand side converges to zero. This implies that $\mathbf{P}(I) = 0$.

SECTION 1.6. Counting

Problem 49. De Méré's puzzle. A six-sided die is rolled three times independently. Which is more likely: a sum of 11 or a sum of 12? (This question was posed by the French nobleman de Méré to his friend Pascal in the 17th century.)

Problem 50. The birthday problem. Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day

during the year, independent of everyone else, and ignore the additional complication presented by leap years (i.e., assume that nobody is born on February 29). What is the probability that each person has a distinct birthday?

Problem 51. An urn contains m red and n white balls.

(a) We draw two balls randomly and simultaneously. Describe the sample space and calculate the probability that the selected balls are of different color, by using two approaches: a counting approach based on the discrete uniform law, and a sequential approach based on the multiplication rule.

(b) We roll a fair 3-sided die whose faces are labeled 1,2,3, and if k comes up, we remove k balls from the urn at random and put them aside. Describe the sample space and calculate the probability that all of the balls drawn are red, using a divide-and-conquer approach and the total probability theorem.

Problem 52. We deal from a well-shuffled 52-card deck. Calculate the probability that the 13th card is the first king to be dealt.

Problem 53. Ninety students, including Joe and Jane, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Joe and Jane end up in the same class?

Problem 54. Twenty distinct cars park in the same parking lot every day. Ten of these cars are US-made, while the other ten are foreign-made. The parking lot has exactly twenty spaces, all in a row, so the cars park side by side. However, the drivers have varying schedules, so the position any car might take on a certain day is random.

(a) In how many different ways can the cars line up?

(b) What is the probability that on a given day, the cars will park in such a way that they alternate (no two US-made are adjacent and no two foreign-made are adjacent)?

Problem 55. Eight rooks are placed in distinct squares of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another, i.e., that there is no row or column with more than one rook.

Problem 56. An academic department offers 8 lower level courses: $\{L_1, L_2, \ldots, L_8\}$ and 10 higher level courses: $\{H_1, H_2, \ldots, H_{10}\}$. A valid curriculum consists of 4 lower level courses, and 3 higher level courses.

(a) How many different curricula are possible?

(b) Suppose that $\{H_1, \ldots, H_5\}$ have L_1 as a prerequisite, and $\{H_6, \ldots H_{10}\}$ have L_2 and L_3 as prerequisites, i.e., any curricula which involve, say, one of $\{H_1, \ldots, H_5\}$ must also include L_1. How many different curricula are there?

Problem 57. How many 6-word sentences can be made using each of the 26 letters of the alphabet exactly once? A word is defined as a nonempty (possibly jibberish) sequence of letters.

Problem 58. We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:

(a) The 7 cards include exactly 3 aces.

(b) The 7 cards include exactly 2 kings.

(c) The probability that the 7 cards include exactly 3 aces, or exactly 2 kings, or both.

Problem 59. A parking lot contains 100 cars, k of which happen to be lemons. We select m of these cars at random and take them for a test drive. Find the probability that n of the cars tested turn out to be lemons.

Problem 60. A well-shuffled 52-card deck is dealt to 4 players. Find the probability that each of the players gets an ace.

Problem 61.* Hypergeometric probabilities. An urn contains n balls, out of which m are red. We select k of the balls at random, without replacement (i.e., selected balls are not put back into the urn before the next selection). What is the probability that i of the selected balls are red?

Solution. The sample space consists of the $\binom{n}{k}$ different ways that we can select k out of the available balls. For the event of interest to occur, we have to select i out of the m red balls, which can be done in $\binom{m}{i}$ ways, and also select $k-i$ out of the $n-m$ balls that are not red, which can be done in $\binom{n-m}{k-i}$ ways. Therefore, the desired probability is

$$\frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}},$$

for $i \geq 0$ satisfying $i \leq m$, $i \leq k$, and $k-i \leq n-m$. For all other i, the probability is zero.

Problem 62.* Correcting the number of permutations for indistinguishable objects. When permuting n objects, some of which are indistinguishable, different permutations may lead to indistinguishable object sequences, so the number of distinguishable object sequences is less than $n!$. For example, there are six permutations of the letters A, B, and C:

ABC, ACB, BAC, BCA, CAB, CBA,

but only three distinguishable sequences that can be formed using the letters A, D, and D:

ADD, DAD, DDA.

(a) Suppose that k out of the n objects are indistinguishable. Show that the number of distinguishable object sequences is $n!/k!$.

(b) Suppose that we have r types of indistinguishable objects, and for each i, k_i objects of type i. Show that the number of distinguishable object sequences is

$$\frac{n!}{k_1!\,k_2!\cdots k_r!}.$$

Solution. (a) Each one of the $n!$ permutations corresponds to $k!$ duplicates which are obtained by permuting the k indistinguishable objects. Thus, the $n!$ permutations can be grouped into $n!/k!$ groups of $k!$ indistinguishable permutations that result in the same object sequence. Therefore, the number of distinguishable object sequences is $n!/k!$. For example, the three letters A, D, and D give the $3! = 6$ permutations

$$\text{ADD, ADD, DAD, DDA, DAD, DDA,}$$

obtained by replacing B and C by D in the permutations of A, B, and C given earlier. However, these 6 permutations can be divided into the $n!/k! = 3!/2! = 3$ groups

$$\{\text{ADD, ADD}\},\ \{\text{DAD, DAD}\},\ \{\text{DDA, DDA}\},$$

each having $k! = 2! = 2$ indistinguishable permutations.

(b) One solution is to extend the argument in (a) above: for each object type i, there are $k_i!$ indistinguishable permutations of the k_i objects. Hence, each permutation belongs to a group of $k_1!\,k_2!\cdots k_r!$ indistinguishable permutations, all of which yield the same object sequence.

An alternative argument goes as follows. Choosing a distinguishable object sequence is the same as starting with n slots and for each i, choosing the k_i slots to be occupied by objects of type i. This is the same as partitioning the set $\{1, \ldots, n\}$ into groups of size $k_1, \ldots, k_r$, and the number of such partitions is given by the multinomial coefficient.