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INTRODUCTION TO PROBABILITY

by

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CHAPTER 7: ADDITIONAL PROBLEMS<sup>†</sup>

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## SECTION 7.1. Some Useful Inequalities

**Problem 1.** Let  $X$  be a random variable and let  $\alpha$  be a positive constant. Show that

$$\mathbf{P}(|X| \geq c) \leq \frac{\mathbf{E}[|X|^\alpha]}{c^\alpha}, \quad \text{for all } c > 0.$$

**Problem 2.**

- (a) Given the information  $\mathbf{E}[X] = 7$  and  $\text{var}(X) = 9$ , use the Chebyshev inequality to find a lower bound for the probability  $\mathbf{P}(4 < X < 10)$ .
- (b) Find the smallest and the largest possible values of the probability  $\mathbf{P}(4 < X < 10)$ , given the mean and variance information from part (a).

**Problem 3.** Investigate whether the Chebyshev inequality is tight. That is, for every  $\mu$ ,  $\sigma$ , and  $c \geq \sigma$ , does there exist a random variable  $X$  with mean  $\mu$  and standard deviation  $\sigma$  such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}?$$

**Problem 4. Chernoff bound for a Poisson random variable.** Let  $X$  be a Poisson random variable with parameter  $\lambda$ .

- (a) Show that for every  $s \geq 0$ , we have

$$\mathbf{P}(X \geq k) \leq e^{\lambda(e^s - 1)} e^{-sk}.$$

- (b) Assuming that  $k > \lambda$ , show that

$$\mathbf{P}(X \geq k) \leq \frac{e^{-\lambda} (e\lambda)^k}{k^k}.$$

## SECTION 7.2. The Weak Law of Large Numbers

**Problem 5.** Bo assumes that  $X$ , the height in meters of any Canadian selected by an equally likely choice among all Canadians, is a random variable with  $\mathbf{E}[X] = h$ . Because Bo is sure that no Canadian is taller than 3 meters, he decides to use 1.5 meters as a conservative value for the standard deviation of  $X$ . To estimate  $h$ , Bo uses the average of the heights of  $n$  Canadians he selects at random.

- (a) In terms of  $h$  and Bo's 1.5 meter bound for the standard deviation of  $X$ , determine the expectation and standard deviation of  $H$ .
- (b) Find as small a value of  $n$  as possible such that the standard deviation of Bo's estimator is guaranteed to be less than 0.01 meters.
- (c) Bo would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of  $n$  that will achieve this objective.

- (d) If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for  $X$ , the height of any Canadian selected at random?

### SECTION 7.3. Convergence in Probability

**Problem 6.** Let  $X_1, X_2, \dots$  be independent, identically distributed random variables with  $\mathbf{E}[X] = 2$  and  $\text{var}(X) = 9$ , and let  $Y_i = X_i/2^i$ . We also define  $T_n$  and  $A_n$  to be the sum and the sample mean, respectively, of the random variables  $Y_1, \dots, Y_n$ .

- Evaluate the mean and variance of  $Y_n$ ,  $T_n$ , and  $A_n$ .
- Does  $Y_n$  converge in probability? If so, to what value?
- Does  $T_n$  converge in probability? If so, to what value?
- Does  $A_n$  converge in probability? If so, to what value?

**Problem 7.** Suppose that a sequence  $X_n$  of random variables satisfies

$$\lim_{n \rightarrow \infty} \mathbf{E}[|X_n - c|^\alpha] = 0,$$

where  $\alpha$  is a positive number. Show that the sequence  $X_n$  converges to  $c$  in probability.

**Problem 8.** Consider a Poisson process, with mean arrival rate  $\lambda = 1$ , and let  $X_n$  be the number of arrivals between time zero and  $n$ . Does  $X_n/n$  converge in probability?

**Problem 9.** Let  $X_1, X_2, \dots$  be independent, identically distributed random variables with (unknown but finite) mean  $\mu$  and positive variance. For  $i = 1, 2, \dots$ , let

$$Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}.$$

- Are the random variables  $Y_i$  independent?
- Are they identically distributed?
- Let

$$M_n = \frac{1}{n} \sum_{i=1}^n Y_i.$$

Show that  $M_n$  converges to  $\mu$  in probability.

**Problem 10.** Let  $X_1, X_2, \dots$  be a sequence of independent random variables that are uniformly distributed between 0 and 1. For every  $n$ , we let  $Y_n$  be the median of the values of  $X_1, X_2, \dots, X_{2n+1}$ . [That is, we order  $X_1, \dots, X_{2n+1}$  in increasing order and let  $Y_n$  be the  $(n+1)$ st element in this ordered sequence.] Show that the sequence  $Y_n$  converges to  $1/2$ , in probability.

### SECTION 7.4. The Central Limit Theorem

**Problem 11.** Uncle Henry has been having trouble keeping his weight constant. In fact, at the end of each week, he notices that his weight has changed by a random

amount, uniformly distributed between  $-0.5$  and  $0.5$  pounds. Assuming that the weight change during any given week is independent of the weight change of any other week, find the probability that Uncle Henry will gain or lose more than 3 pounds in the next 50 weeks.

**Problem 12.** On any given flight, an airline's goal is to fill the plane as much as possible, without overbooking. If, on average, 10% of customers cancel their tickets, all independently of each other, what is the probability that a particular flight will be overbooked if the airline sells 320 tickets, for a plane that has maximum capacity 300 people? What is the probability that a plane with maximum capacity 150 people will be overbooked if the airline sells 160 tickets?

**Problem 13.** Alex puts some pennies into a piggy-bank each day. The number of pennies added on any given day is equally likely to be 1, 2, 3, 4, 5, or 6, and is independent from day to day. Find an approximation to the probability that it takes at least 80 days to collect 3 dollars.

**Problem 14.** Let  $X_1, \dots, X_{10}$  be independent random variables, uniformly distributed over the unit interval  $[0, 1]$ .

- (a) Estimate  $\mathbf{P}(X_1 + \dots + X_{10} \geq 7)$  using the Markov inequality.
- (b) Repeat part (a) using the Chebyshev inequality.
- (c) Repeat part (a) using the central limit theorem.

**Problem 15.** Let  $S_n$  be the number of successes in  $n$  independent Bernoulli trials, where the probability of success in each trial is  $p = 1/2$ . Provide a numerical value for the limit as  $n$  tends to infinity for each of the following three expressions.

- (a)  $\mathbf{P}\left(\frac{n}{2} - 10 \leq S_n \leq \frac{n}{2} + 10\right)$ .
- (b)  $\mathbf{P}\left(\frac{n}{2} - \frac{n}{10} \leq S_n \leq \frac{n}{2} + \frac{n}{10}\right)$ .
- (c)  $\mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{2} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2}\right)$ .

**Problem 16.** The adult population of Nowhereville consists of 300 males and 196 females. Each male (respectively, female) has a probability of 0.4 (respectively, 0.5) of casting a vote in the local elections, independently of everyone else. Find a good numerical approximation for the probability that more males than females cast a vote.

## SECTION 7.5. The Strong Law of Large Numbers

**Problem 17.** Let  $X_1, X_2, \dots$  be a sequence of independent identically distributed random variables with finite mean and variance. Show that the sequence  $Y_n = X_n/n$  converges to zero, with probability 1.

**Problem 18.** Let  $X_1, X_2, \dots$  be independent identically distributed random variables with mean 5, variance 9, and such that  $\mathbf{P}(X_n = 0) > 0$ . For each of the following ways of defining  $Z_n$ , determine whether the sequence  $Z_n$  converges with probability 1, and if it does, identify the limit.

- (a)  $Z_n = (X_1 + \dots + X_n)/n$ .
- (b)  $Z_n = (X_1 + \dots + X_n - 5n)/\sqrt{n}$ .

(c)  $Z_n = (X_1^2 + \cdots + X_n^2)/n$ .

(d)  $Z_n = X_1 X_2 \cdots X_n$ .

(e)  $Z_n = (X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n)/n$ .

**Problem 19.** The fortune  $X_n$  of a gambler evolves as  $X_n = Z_n X_{n-1}$ , where the  $Z_n$  are independent identically distributed random variables with PMF

$$p_Z(z) = \begin{cases} 1/3, & \text{for } z = 3, \\ 2/3, & \text{for } z = 1/3. \end{cases}$$

Assume that  $X_0 = 1$ .

(a) Show that the expected fortune  $\mathbf{E}[X_n]$  converges to infinity as  $n$  increases.

(b) Show that the actual fortune  $X_n$  converges to 0 with probability 1.

**Problem 20.** Consider a Poisson process with arrival rate  $\lambda = 1$ . We divide the infinite time axis into disjoint intervals of length 1, 1/2, 1/3, 1/4, etc., so that the  $k$ th interval has length  $1/k$ . These intervals are getting smaller, but their total length is infinite because  $\sum_{k=1}^{\infty} 1/k = \infty$ . Let  $Z_k$  be the number of arrivals during the  $k$ th interval. Does the sequence  $Z_k$  converge in probability (and if so, to what)? Does the sequence  $Z_k$  converge with probability 1 (and if so, to what)? Provide a brief justification.