INTRODUCTION TO PROBABILITY

by

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CHAPTER 5: ADDITIONAL PROBLEMS†

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† Problems marked with "[D]" are from "Fundamentals of Applied Probability", by Alvin Drake, and are included here with the author's permission.

SECTION 5.1. The Bernoulli Process

Problem 1. We are given a coin for which the probability of heads is p $(0 < p < 1)$ and the probability of tails is $1 - p$. Consider a sequence of independent flips of the coin.

(a) Let Y be the number of flips up to and including the flip on which the first head occurs. Write down the PMF of Y.

(b) Let X be the number of heads that occur on any particular flip. Write down $\mathbf{E}[X]$ and $\text{var}(X)$.

(c) Let K be the number of heads that occur on the first n flips of the coin. Determine the PMF, mean, and variance of K.

(d) Given that a total of exactly six heads resulted from the first nine flips, what is the conditional probability that both the first and seventh flips were tails?

(e) Let H be the number of heads that occur on the first twenty flips. Let C be the event that a total of exactly ten heads resulted from the first eighteen flips. Find $\mathbf{E}[H \mid C]$ and the conditional variance $\text{var}(H \mid C)$.

Problem 2. At each trial of a game, Don and Greg flip biased coins, simultaneously but independently. For each trial, the probability of heads is p_D and p_G for Don and Greg, respectively.

(a) Given that the flips on a particular trial resulted in 2 heads, find the PMF of the number of additional trials up to and including the next trial on which 2 heads result.

(b) Given that the flips on a particular trial resulted in at least one head, find the probability that Don flipped a head on that trial.

(c) Starting from a trial on which no heads result, find the probability that Don's next flip of a head will occur before Greg's next flip of a head.

(d) Given that Don receives \$$d$ for each head he flips, and Greg receives \$$g$ for each head he flips, find the transform associated with the total amount of money earned by the two players during the first n trials.

Problem 3. **[D]** To cross a single lane of moving traffic, we require at least a duration d. Successive car interarrival times are independently and identically distributed with probability density function $f_T(t)$. If an interval between successive cars is longer than d, we say that the interval represents a single opportunity to cross the lane. Assume that car lengths are small relative to intercar spacing and that our experiment begins the instant after the zeroth car goes by. Determine, in as simple form as possible, expressions for the probability that:

(a) We can cross for the first time just before the nth car goes by.

(b) We shall have had exactly m opportunities by the instant the nth car goes by.

(c) The occurrence of the mth opportunity is immediately followed by the arrival of the nth car.

Problem 4. Let Y_{17} be a Pascal random variable of order 17. Find the numerical values of a and b in the equation

$$\sum_{l=42}^{\infty} p_{Y_{17}}(l) = \sum_{k=0}^{a} \binom{b}{k} p^k (1-p)^{(b-k)},$$

and explain.

Problem 5. **[D]** Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered **a**nd a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

(a) Determine the probability that Fred gives away his first sample on his third call.

(b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

(c) Determine the probability that he gives away his second sample on his fifth call.

(d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

(e) We will say that Fred "needs a new supply" immediately **a**fter the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

(f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

Problem 6. Alice and Bob alternate playing at the casino table. (Alice starts and plays at odd times $i = 1, 3, \ldots$; Bob plays at even times $i = 2, 4, \ldots$.) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$p_G(g) = \begin{cases} 1/3, & g = -2, \\ 1/2, & g = 1, \\ 1/6, & g = 3. \end{cases}$$

Assume that the net gains at different times are independent. We refer to an outcome of -2 as a "loss," and an outcome of 1 or 3 as a "win."

(a) They keep gambling until the first time where a loss by Bob immediately follows a loss by Alice. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Alice and then one by Bob.)

(b) Write down an expression for the transform of the net gain of Alice up to the time of the first loss by Bob.

(c) Write down the PMF for Z, defined as the time at which Bob has his third loss.

(d) Let N be the number of rounds until each one of them has won at least once. Find $\mathbf{E}[N]$.

Problem 7. Each night, the probability of a robbery attempt at the local warehouse is 1/5. A robbery attempt is successful with probability 3/4, independent of other nights. After any particular successful robbery, the robber celebrates by taking off either the next 2 or 4 nights (with equal probability), during which time there will be no robbery attempts. After that, the robber returns to his original routine.

(a) Let K be the number of robbery attempts up to (and including) the first successful robbery. Find the PMF of K.

(b) Let D be the number of days until (and including) the second successful robbery, including the days of celebration after the first robbery. Find the PMF of D, or its transform (whichever you find more convenient).

During a successful robbery, the robber steals a random number of candy bars, which is 1, 2, or 3, with equal probabilities. This number is independent for each successful robbery and independent of everything else. No candy bars are stolen in unsuccessful robberies.

(c) Let S be the number of candy bars collected in two successful robberies. Find the PMF of S.

(d) Let Q be the number of candy bars collected in ten robbery attempts (whether successful or not). Find the PMF of Q, or its transform, whichever is easier. Find the expectation and the variance of Q.

Problem 8. A particular medical operation proves fatal in 1% of the cases. Find an approximation to the probability that there will be at least 2 fatalities in 200 operations.

Problem 9. You drive to work 5 days a week for a full year (50 weeks), and on any given day, you get a traffic ticket with probability $p = 0.02$, independently of other days. Let X be the total number of tickets you get in the year.

(a) What is the probability that the number of tickets you get is exactly equal to the expected value of X?

(b) Calculate approximately the probability in (a), using a Poisson approximation.

(c) The fine for a ticket is \$10, or \$20, or \$50, with respective probabilities 0.5, 0.3, and 0.2, and independently of other tickets. Find the mean and variance of the total amount you pay for traffic tickets during the year?

(d) Suppose you do not know the probability p of getting a ticket, but you got 5 tickets during the year, and you estimate p by the sample mean

$$\hat{P} = \frac{5}{250}.$$

What is the range of possible values of p assuming that the difference between p and the sample mean $\hat{P}$ is within 5 times the standard deviation of the sample mean? (See Example 2.21 in the text for more detail on the use and properties of the sample mean.)

SECTION 5.2. The Poisson Process

Problem 10. A train bridge is constructed across a wide river. Trains arrive at the bridge according to a Poisson process of rate $\lambda = 3$ per day.

(a) If a train arrives on day 0, find the probability that there will be no trains on days 1, 2, and 3.

(b) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4th day.

(c) Find the probability that it takes more than 2 days for the 5th train to arrive at the bridge.

Problem 11. An amateur criminal is contemplating shoplifting from a store. Police officers walk by the store according to a Poisson process of rate λ per minute. If an officer walks by while the crime is in progress, the criminal will be caught.

(a) If it takes the criminal t seconds to commit the crime, find the probability that the criminal will be caught.

(b) Repeat part (a) under the new assumption that the criminal will only be caught if two police officers happen to walk by while the crime is in progress.

Problem 12. The MIT soccer team needs at least 8 players to avoid forfeiting a game. Assume that each player has some chance of getting injured for the season, but her playing lifetime for a given season is exponentially distributed with parameter λ. For simplicity, assume that the coach insists on only playing 8 players at a time, and then replaces a player as soon as she gets hurt. Find:

(a) The expected time until the first substitution.

(b) The distribution of total time the team can play in a season, given that there are n women on the team.

Problem 13. A certain police officer stops cars for speeding. The number of red sports cars she stops in one hour is a Poisson process with rate 4, while the number of other cars she stops is a Poisson process with rate 1. Assume that these two processes are independent of each other. Find the probability that this police officer stops at least 2 ordinary cars before she stops 3 red sports cars.

Problem 14. Consider two independent Poisson processes, with arrival rates α and β, respectively. Determine:

(a) The probability q that the next three arrivals come from the same process.

(b) The PMF of N, the number of arrivals from the first process that occur before the fourth arrival from the second process.

Problem 15. Suppose the waiting time until the next bus at a particular bus stop is exponentially distributed with parameter $\lambda = 1/15$. Suppose that a bus pulls out just as you arrive at the stop. Find the probability that:

(a) You wait more than 15 minutes for a bus.

(b) You wait between 15 and 30 minutes for a bus.

(c) What are the probabilities in (a) and (b) assuming the bus left 5 minutes before you arrive?

Problem 16. A phone at a telephone exchange rings according to a Poisson process of rate λ. If 3 calls arrive in the first ninety minutes, find:

(a) The probability that all 3 calls arrived in the first 30 minutes.

(b) The probability that at least one arrived in the first 30 minutes.

Problem 17. The time to finish a problem set is exponentially distributed with parameter $\lambda = 1/2$.

(a) Find the probability that a particular problem set takes more than 2 hours to finish.

(b) Given that you have been working on a problem set already for 7 hours, find the probability that this problem set will take more than 9 hours total (i.e., two hours more).

Problem 18. Based on your understanding of the Poisson process, determine the numerical values of a and b in the following expression and explain your reasoning.

$$\int_t^\infty \frac{\lambda^6 \tau^5 e^{-\lambda\tau}}{5!}\, d\tau = \sum_{k=a}^{b} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

Problem 19. **[D]** A woman is seated beside a conveyor belt, and her job is to remove certain items from the belt. She has a narrow line of vision and can get these items only when they are right in front of her. She has noted that the probability that exactly k of her items will arrive in a minute is given by

$$p_K(k) = \frac{2^k e^{-2}}{k!}, \qquad k = 0, 1, 2, \ldots,$$

and she assumes that the arrivals of her items constitute a Poisson process.

(a) If she wishes to sneak out to have a beer but will not allow the expected value of the number of items she misses to be greater than 5, how much time may she take?

(b) If she leaves for two minutes, what is the probability that she will miss exactly two items the first minute and exactly one item the second minute?

(c) If she leaves for two minutes, what is the probability that she will miss a total of exactly three items?

(d) The union has installed a bell which rings once a minute with precisely one-minute intervals between gongs. If, between two successive gongs, more than three items come along the belt, she will handle only three of them properly and will destroy the rest. Under this system, what is the expected fraction of items that will be destroyed?

Problem 20. **[D]** Arrivals of certain events at points in time are known to constitute a Poisson process, but it is not known which of two possible values of λ, the average arrival rate, describes the process. Our a priori estimate is that $\lambda = 2$ or $\lambda = 4$ with equal probability. We observe the process for t units of time and observe exactly k arrivals. Given this information, determine the conditional probability that $\lambda = 2$. Check to see whether or not your answer is reasonable for some simple limiting values for k and t.

Problem 21. **[D]** Let $K_1, , K_2, \ldots$ be independent identically distributed geometric random variables. Random variable R_i is defined by

$$R_i = \sum_{j=1}^{i} K_i, \qquad i = 1, 2, \ldots.$$

If we eliminate arrivals number $R_1, R_2, \ldots$ in a Poisson process, do the remaining arrivals constitute a Poisson process?

Problem 22. Determine, in an efficient manner (without using integration by parts), the fourth moment of a continuous random variable described by the probability density function

$$f_X(x) = \frac{4^3 x^2 e^{-4x}}{2}, \qquad x \geq 0.$$

Problem 23. A room has two lamps that use bulbs of type A and B, respectively. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

$$\text{for type-A Bulbs: } f_X(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{for type-B Bulbs: } f_X(x) = \begin{cases} 3e^{-3x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Both lamps are lit at time zero. Whenever a bulb is burned out it is immediately replaced by a new bulb.

(a) What is the expected value of the number of type-B bulb failures until time t?

(b) What is the PDF of the time until the first failure of *either* bulb type?

(c) Find the expected value and variance of the time until the third failure of a type-B bulb.

(d) Suppose that a type-A bulb has just failed. How long do we expect to wait until a subsequent type-B bulb failure?

Problem 24. **[D]** Dave is taking a multiple-choice exam. You may assume that the number of questions is infinite. Simultaneously, but independently, his conscious and subconscious faculties are generating answers for him, each in a Poisson manner. (His conscious and subconscious are always working on different questions.) Conscious responses are generated at a rate of λ_c responses per minute. Subconscious responses are generated at a rate of λ_s responses per minute. Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct. Dave responds only once to each question, and you can assume that his time for recording these conscious and subconscious responses is negligible.

(a) Determine $p_K(k)$, the probability mass function for the number of conscious responses Dave makes in an interval of t minutes.

(b) If we pick any question to which Dave has responded, what is the probability that his answer to that question:

(i) Represents a conscious response.

(ii) Represents a conscious correct response.

(c) If we pick an interval of t minutes, what is the probability that in that interval Dave will make exactly r conscious responses *and* exactly s subconscious responses?

(d) Determine the transform for the probability density function for random variable X, where X is the time from the start of the exam until Dave makes his first conscious response which is preceded by at least one subconscious response.

(e) Determine the probability mass function for the total number of responses up to and including his third conscious response.

(f) The papers are to be collected as soon as Dave has completed exactly n responses. Determine:

(i) The expected number of questions he will answer correctly

(ii) The probability mass function for L, the number of questions he answers correctly.

(g) Repeat part (f) for the case in which the exam papers are to be collected at the end of a fixed interval of t minutes.

Problem 25. There are two types of calls to the MIT Campus Patrol. Type A calls (distress calls) arrive as a Poisson process with rate λ_A. Type B calls (professors who have lost their keys) arrive as an independent Poisson process with rate λ_B. Let us fix t to be 12 o'clock.

(a) What is the expected length of the interval that t belongs to? (That is, the interval from the last event before t until the first event after t.)

(b) What is the probability that t belongs to an AA interval? (That is, the first event before, as well as the first event after time t are both of type A.)

(c) Let c be a constant. What is the probability that between t and $t + c$, we have exactly two events, one of type A, followed by one of type B?

Problem 26. **[D]** The interarrival times for cars passing a checkpoint are independent random variables with PDF

$$f_T(t) = \begin{cases} 2e^{-2t}, & t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where the interarrival times are measured in minutes. The successive values of the durations of these interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

(a) Determine the mean and the third moment of the interarrival times.

(b) Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.

(c) Determine the PDF, the expected value, and the transform for the total time required to use up the first dozen computer cards.

(d) Consider the following two experiments:

(i) Pick a card at random from a group of completed cards and note the total time, U, the card was in service. Find $\mathbf{E}[U]$ and σ_U^2.

(ii) Come to the corner at a certain time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time V. Determine $\mathbf{E}[V]$ and σ_V^2.

(e) Given that the computer card presently in use contains exactly two entries and also that it has been in service for exactly 0.5 minute, determine the PDF for the remaining time until the card is completed.

Problem 27.

(a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

(b) Now, and for the rest of this problem, suppose that the shuttles are not operating on a deterministic schedule, but rather their interdeparture times are exponentially distributed with rate μ per hour, and independent of the process of passenger arrivals. Find the PMF of the number shuttle departures in one hour.

(c) Let us define an "event" in the airport to be either the arrival of a passenger, or the departure of a plane. Find the expected number of "events" that occur in one hour.

(d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

(e) Find the PMF of the number of people on a shuttle.

Problem 28. Type A, B, and C items are placed in a common buffer, each type arriving as part of an independent Poisson process with average arrival rates, respectively, of a, b, and c items per minute.

For the first four parts of this problem, assume the buffer is discharged immediately whenever it contains a total of ten items.

(a) What is the probability that, of the first ten items to arrive at the buffer, only the first and one other are type A?

(b) What is the probability that any particular discharge of the buffer contains five times as many type A items as type B items?

(c) Determine the PDF, expectation, and variance of the total time between consecutive discharges of the buffer.

(d) Determine the probability that during a particular five minute interval there exactly two arrivals of each type.

For the rest of this problem, a different rule is used for discharging the buffer: namely, the buffer is discharged immediately whenever it contains a total of three type A items.

(e) Determine the PDF, expectation, and variance of the total time between consecutive discharges of the buffer.

(f) For an observer arriving at a random time, long after the process began, obtain the PDFs of:

  (i) U, the time until the arrival of the next item at the buffer input

  (ii) V, the time until the next discharge of the buffer

Problem 29. Let T_1, T_2 (respectively, S) be exponential random variables with parameter λ (respectively, μ). We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1 + T_2, S\}$.

Problem 30. Let Y be exponentially distributed with parameter λ_1. Let Z_k be Erlang of order k. with parameter λ_2. Assume that Y and Z_k are independent. Let $M_k = \max\{Y, Z_k\}$. Find a recursive formula for $\mathbf{E}[M_k]$, in terms of $E[M_{k-1}]$.

Problem 31. Consider the random variable $Z = X - Y$, where X and Y are independent and exponentially distributed with parameter λ.

(a) Find the PDF of Z by conditioning on the events $\{X \leq Y\}$ and $\{X < Y\}$, and using an interpretation in terms of Poisson arrivals.

(b) Repeat part (a) for the case where X and Y are independent and exponentially distributed, but with different parameters λ_X and λ_Y.

Problem 32. **[D]** Al makes cigars, placing each cigar on a constant-velocity conveyor belt as soon as it is finished. Bo packs the cigars into boxes of four cigars each, placing each box back on the belt as soon as it is filled. The time Al takes to construct any particular cigar is, believe it or not, an independent exponential random variable with an expected value of five minutes.

(a) Let K be the number of cigars that Al makes in t minutes. Determine $P_A(k, t)$, the probability that Al makes exactly k cigars in t minutes. Determine the mean and variance of K as a function of t.

(b) Determine the probability density function $f_T(t)$, where T is the interarrival time (measured in minutes) between placing successive cigars on the conveyor belt.

(c) Determine $P_B(r, t)$, the probability that Bo places exactly r boxes of cigars back on the belt during an interval of t minutes.

(d) Determine the probability density function $f_S(s)$ where S is the interarrival time (measured in minutes) between placing successive boxes of cigars on the conveyor belt.

(e) If we arrive at a certain point in time, long after the process began, determine the PDF $f_R(r)$, where R is the duration of our wait until we see a box of cigars.

Problem 33. **[D]** All ships travel at the same speed through a wide canal. Eastbound ships arrive as a Poisson process with an arrival rate of λ_E ships per day. Westbound ships arrive as an independent Poisson process with an arrival rate of λ_W ships per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass. Each ship takes t days to traverse the canal.

(a) What is the probability that the next ship passing by the indicator causes it to change its direction?

(b) What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

(c) If we begin observing at an arbitrary time, determine the probability mass function of the total number of ships we observe up to and including the seventh eastbound ship we see.

(d) If we begin observing at an arbitrary time, determine the PDF of the time until we see the seventh eastbound ship.

(e) Given that the pointer is pointing west:

  (i) What is the probability that the next ship to pass it will be westbound?

  (ii) What is the PDF of the remaining time until the pointer changes direction?

Problem 34. We are given the following statistics about the number of children in a typical family in a small village. There are 100 families. 10 families have no children, 40 have 1, 30 have 2, 10 have 3, 10 have 4.

(a) If you pick a family at random, what is the expected number of children in that family?

(b) If you pick a child at random (each child is equally likely), what is the expected number of children in that child's family (including the picked child)?

(c) Generalize your approach from part (b) to the case where a fraction p_k of the families has k children, and provide a formula.

Problem 35. Consider a Poisson process of rate λ. Let N be the number of arrivals in $(0, t]$, and let M be the number of arrivals in $[0, t + s]$.

(a) Find the joint PMF of N and M.

(b) Find $\mathbf{E}[NM]$.